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CONTENTS

Analogues For Arithmetical Functions of the Elementary Transcendental Functions

by E. T. BELL

Combinatorial analysis has been used[1] to obtain identities for the kind of numerical functions (such as the sum and the number of divisors of an integer) which arise directly from the fundamental theorem of arithmetic for the rational integers, and which are independent of order (greater, less) relations. It seems more natural, however, to proceed in the opposite direction, from theorems in a fully developed algebra[2] of numerical functions to their unique correspondents in an algebra of distribution functions. Each theorem in the first algebra then implies and is implied by a unique combinatorial correspondent in the second. Before the relevant isomorphism can be established, it is necessary to recall (Section 1) certain details of the algebra of numerical functions mentioned, and then (Section 2), in that algebra, to define analogues of the exponential and logarithmic functions, and to show that these functions satisfy the same defining equations as those for the corresponding analytic functions, although the interpretations of the functions are basically different. This will be done in the present paper; a sequel will give the applications to combinatorial analysis. Section 3 contains simple examples of some of the general results in Section 2.

1. *Addition, multiplication of numerical functions.* There are many ways of defining the rational operations for numerical functions so that commutative groups, rings, fields result. The way chosen is one of the simplest.

(1.1) If for all (finite) positive integer values n of x, $f(x)$ is

(a) uniform (single-valued),

(b) a finite (real or complex) number,

$f(x)$ is a *numerical function of* x. We shall write f for $f(x)$, and call f a *numerical function.*

(1.2) If f is a numerical function such that $f(1) \neq 0$, f is *regular*; if $f(1) = 0$, f is *irregular.*

(1.3) If f is regular, we shall take $f(1) = 1$, provided $f(1)$ is not otherwise defined. For example, the arithmetical definition of Mobius' μ includes the convention $\mu(1) = 1$.

(1.4) If f, g are numerical functions such that

$$f(n)=g(n), \quad n=1,2,3,\cdots,$$

this is written $f=g$, and f, g are defined to be *equal*.

Obviously this equality has all the customary postulated properties of abstract equality.

(1.5) The *unit* numerical functions η is defined by

$$\eta(1)=1, \quad \eta(n)=0, \quad n=2,3,4,\cdots,$$

and the zero numerical function ω by

$$\omega(n)=0, \quad n=1,2,3,\cdots.$$

(1.6) The *sum* h of the numerical functions f, g is defined by

$$h(n)=f(n)+g(n), \quad n=1,2,3,\cdots,$$

and this is written $h=f+g$.

(1.7) If c is a real or complex number, and f, g are numerical functions such that

$$cf(n)=g(n), \quad n=1,2,3,\cdots,$$

then, by (1.4), $cf=g$. In the same way fc is defined, and $cf=fc$, either of which is the *scalar product* of c, f ; also $c_1(c_2f)=(c_1c_2)f$, where c_1, c_2 are *scalars* (real or complex numbers).

(1.8) If f, g, h are numerical functions such that

$$\sum f(n_1)g(n_2)=h(n), \qquad n=1,2,3,\cdots,$$

where the sum refers to all pairs (n_1, n_2) of positive integers n_1, n_2 such that $n_1n_2=n$, h is the *product* (unqualified) of f, g and this is written $fg=h$. Evidently, $\eta f=f$.

(1.9) From (1.8) it follows that if $f^{(1)},\cdots,f^{(t)}$ are numerical functions, $f^{(1)}\cdots f^{(t)}=h$ is equivalent to

$$\sum f^{(1)}(n_1)\cdots f^{(t)}(n_t)=h(n), \quad n=1,2,3,\cdots,$$

the summation referring to all $(n_1,\cdots,n_t)$ where $n_1,\cdots,n_t$ are positive integers such that $n_1\cdots n_t=n$.

(1.10) *Theorem.*[3] If f is regular (as defined in (1.3)), there is a unique numerical function g, also regular, such that $fg=\eta$, where η is as in (1.5). We write $g\equiv f^{-1}\equiv \eta/f$ and call f^{-1} the *reciprocal* of f.

If in (1.9), $f^{(1)}=\cdots=f^{(t)}=f$, we write $h=f^t$, and define $f^0\equiv\eta$. The reciprocal of f^t is $(f^{-1})^t$, which will be written f^{-t}, or η/f^t ; $(f^{-1})^t=(f^t)^{-1}$; $f^tf^{-t}=\eta=f^0$.

(1.11) *Theorem.* (a) The set of all regular numerical functions defined in (1.1) is a commutative group under multiplication as in (1.8); the identity of the group is η. (b) For addition as in (1.6) the set of all scalar products as defined in (1.7) is a commutative group with the identity ω as in (1.5). (c) The set of all scalar products as in (1.7) is commutative ring under multiplication, as in (1.8) and addition as in (1.11) (b).

The G. C. D. of the integers m, n, is denoted as usual by (m, n). If m, n are coprime, $(m, n)=1$.

If the numerical function f is such that

$$f(mn)=f(m)\,f(n)$$

for all positive integers, m, n such that $(m, n)=1$, f is called *factorable.* (The term *multiplicative* has also been used, but is of considerably later origin.) It follows from the definition that $f(1)=1$ if f is factorable.

(1.12) *Theorem.* The set of all factorable numerical functions is a subgroup under multiplication as defined in (1.8) of the group in (1.11) (a).

(1.13) In connection with the algebra summarized in (1.11), (1.12), another type of product frequently occurs, the *absolute product* $|fg|$ of f, g, defined by

$$|fg|(n)=f(n)\ g(n), \quad n=1,2,3,\cdots .$$

A further type of multiplication is immediately suggested when the foregoing algebra is tentatively extended to infinite processes with addition and multiplication as (1.11). For example, if ζ is defined by $\zeta(n)=1$, $n=1,2,3,\cdots$, and f by

$$f(n)\equiv(\zeta/1+\zeta^2/2+\zeta^3/3+\cdots)(n)$$

this f violates (b) in (1.1), and therefore is not a numerical function. A satisfactory supplementary type of multiplication presents itself in the actual applications of the algebra to combinatorial analysis. This will be defined next. Although not the only one possible, it appears to be the simplest.

2. *Restricted multiplication.*[4] With η as in (1.5), t a non-negative integer, and $f(1)=1$, the function f_t is defined by

$$f_0=\eta; \quad f_t=(f-\eta)^t, \quad f_t(1)=0, \quad t>0.$$

It follows that

$$f_0(1)=1, \quad f_0(n)=0, \quad n>1; \quad >f_0=f^0=\eta;$$

and for integers $t>0$, $n>1$,

$$f_t(n)=\sum f(n_1)\cdots f(n_t)=\sum_{s=0}^{t}(-1)^s{}_tC_s f^{t-s}(n),$$

where $\sum$ refers to all $(n_1,\cdots,n_t)$ such that $n=n_1\cdots n_t$, $n_1>1,\cdots,n_t>1$; $f^t, f^{t-1}\cdots, f^0$ are powers as in (1.10); ${}_tC_s$ is the coefficient of x^s in $(1+x)^t$. For an obvious reason, f_t is called the *restricted t^{th} power* of f, in contrast to f^t, which is unrestricted. If c is a scalar (real or complex number), the scalar product $c(f-\eta)^t$, and hence cf_t, is defined, from (1.7); and $c_1(c_2f_t)=c_1c_2f_t$.

An equality of the form

$$g=c_0f_0+c_1f_1+c_2f_2+\cdots,$$

where $c_0, c_1, c_2,\cdots$ are scalars, means

$$g(n)=c_0f_0(n)+c_1f_1(n)+c_2f_2(n)+\cdots,\quad n=1,2,3,\cdots.$$

Since for each n there is a least integer $t(n)\equiv t\geq 0$ such that $f_r(n)=0$, $r>t$ (from the definition of f_t), the foregoing equalities may be written

$$g=\sum_{s=0}^{\infty}c_sf_s,\quad g(n)=\sum_{s=0}^{\infty}c_sf_s(n),\quad n=1,2,3,\cdots.$$

The convenient use of ∞ as the upper limit of the summations can be circumvented, if desired, by replacing ∞ by N, where N is an integer equal to the greatest of the integers n occurring in a given set of formulas involving restricted powers. Either usage amounts to asserting the formulas for arbitrary positive integers. It is important only to note that, for any integer $n>0$, g, $g(n)$ as above defined are finite series, and hence that these satisfy (1.1)(b).

(2.1) *Theorem.* If t is a positive, zero, or negative integer, and $f(1)=1$,

$$f^t=\sum_{s=0}^{\infty}{}_tC_sf_s,$$

where ${}_tC_s$ is the coefficient of x^s in $(1+x)^t$. (For f^t, $t<0$, see (1.10)).

This may be proved from the definitions, and likewise for further theorems in this section. But as the proofs of all are immediate by the method of generators,[5] proofs from the definitions need not be given. The restriction $f(1)=1$ is inessential; $f(1)\neq 0$ suffices, that is, f is regular.

(2.2) *Theorem.* With $f(1)=1$, and $f_r f_s$ the product of f_r, f_s as defined in (1.8).

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}{}_tC_r\,{}_{-t}C_sf_rf_s=\eta.$$

(2.3) If f is any numerical function such that $f(1) = 1$, the functions (Lf), (Ef), or simply, Lf, Ef, are defined by

$$Lf(n) \equiv \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} f_s(n), \quad n = 1,2,3, \cdots ,$$

$$Ef(n) \equiv \sum_{s=0}^{\infty} \frac{1}{s!} f_s(n), \quad n = 1,2,3, \cdots ,$$

which are equivalent, respectively, to

$$Lf \equiv \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} f_s, \qquad Ef \equiv \sum_{s=0}^{\infty} \frac{1}{s!} f_s.$$

The first is the analogue of $\log(1+x)$, $|x| < 1$; the second, of e^y, $y < \infty$. For any integer n the series terminate after a finite of terms (depending on n). Lf is a new function, (Lf), derived from f as indicated and similarly for Ef. These being themselves numerical functions, $(E(Lf))$, $(L(Ef))$ are derived from (Lf), (Ef). The double parentheses may be dropped:

$$ELf = \sum_{s=0}^{\infty} \frac{1}{s!} (Lf)_s, \qquad LEf = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} (Ef)_s.$$

(2.4) *Theorem.* $(Ef)_1 = (Ef)(1) = 1$;

$$(Lf)_0(1) = 1; \quad (Lf)_1(1) = (Lf)(1) = 0.$$

These follow from: $f(1) = 1$ (assumed); $f_0 = \eta$, $f_0(1) = \eta(1) =$; $f_0(n) = \eta(n) = 0$, $n > 1$; the definition of g_s for any numerical function $g1$; and the definitions of Lf, Ef.

(2.5) *Theorem.* $ELf = LEf = f$;

$$LEf(1) = ELf(1) = 1 \quad [= f(1)].$$

To verify the last two:

$$(L(Ef))(1) = \left[\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} (Ef)_s\right] (1),$$

$$= (Ef)_1(1) + (Ef)_2(1) + \cdots ,$$

and only the first term survives, $(Ef)_1(1) = 1$.

$$(E(Lf))(1) = \left[\sum_{s=0}^{\infty} \frac{1}{s!} (Lf)_s\right] (1),$$

$$= (Lf)_0(1) + (Lf)_1(1) + \cdots ,$$

and again only the first term survives, $(Lf)_0(1)=1$.

(2.6) *Theorem.* If $f(1)=g(1)=1$, and $f+g$, fg are as in (1.6), (1.8),

$$E(f+g)=Ef\,Eg,\quad L(fg)=Lf+Lg,$$

where $Ef\,Eg$ is a product as in (1.8), and $Lf+Lg$ a sum as in (1.6).

(2.7) *Theorem.* If $f(1)=g(1)=\cdots=k(1)=1$, and $a, b,\cdots, l$ are (positive, zero, or negative) integers.

$$E(af+bg+\cdots+lk)=Eaf\,Ebg\cdots Elk,$$

$$L(f^a g^b\cdots k^l)=aLf+bLg+\cdots+lLk,$$

in the first of which $af,\cdots$ are scalar products as in (1.7), and in the second, $f^a,\cdots$ are powers as in (1.9), and $aLf,\cdots$ scalar products. This includes (2.6), but is readily derived from it.

By (1.11), $\eta f=f$, $\omega+f=f$. It is easily seen from the definitions that $L\eta=0$, $E\omega=\eta$. Thus, from the above,

$$L\eta f=L\eta+Lf=Lf,\quad E(\omega+f)=E(\omega)Ef=Ef,$$

as should be so if the analogies are exact.

(2.71) If $f^{(j)}$, $j=1,\cdots,s$ are numerical functions such that $f^{(j)}(1)=1$, the *restricted product* $(f^{(1)}\cdots f^{(s)})$ of $f^{(1)},\cdots,f^{(s)}$ is defined by

$$(f^{(1)}\cdots f^{(s)})(n)=\sum f^{(1)}(n_1)\cdots f^{(s)}(n_s),\quad n=1,2,3,\cdots,$$

where the summation refers to all $(n_1,\cdots,n_s)$ such that $n=n_1\cdots n_s$, where $n_1,\cdots,n_s$ are integers >1.

(2.8) *Theorem*[4]

$$(f^{(1)}\cdots f^{(s)})=[f^{(1)}-\eta]\cdots[f^{(s)}-\eta],$$

where the product on the right is to be distributed and interpreted as in (1.11).

For example, since $\eta=\eta^2=\cdots$,

$$(fg)=fg-(f+g)\eta+\eta,$$

$$(fgh)=fgh-(fg+fh+gh)\eta+(f+g+h)\eta-\eta,$$

and so on.

(2.9) *Theorem*

$$(f^{(1)}\cdots f^{(s)})(g^{(1)}\cdots g^{(t)})=(f^{(1)}\cdots f^{(s)}g^{(1)}\cdots g^{(t)}),$$

the product on the left being as in (1.8), and similarly, by (1.9), for a product of any number of restricted products.

This is evident from (2.8).

3. *Analogues for the circular functions.* If Sf, Cf are defined by

$$Sf(n) \equiv \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} f_{2s+1}(n),$$

$$Cf(n) \equiv \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s)!} f_{2s}(n),$$

and if $(Sf)^2$, $(Cf)^2$ are squares as defined in (1.9), then

$$(Sf)^2+(Cf)^2=\eta.$$

Thus Sf, Cf are analogues of sin, cos. The addition theorems, etc., for the circular functions have the expected analogues. For example, if $h=f+g$,

$$Sh=Sf\ Cg+Cf\ Sg, \quad Ch=Cf\ Cg-Sf\ Sg,$$

where $Sf\ Sg$, etc., are products as defined in (1.8). The remaining circular functions also have corresponding analogues.

To obtain the analogues of $\log(1+x)$, e^x, $\sin x$, $\cos x$, we replace x by f in the respective Maclaurin series for these, and replace f^s in the results by f_s, noting that $f^0=f_0=\eta$. The scalar 1 is replaced by η. If Ff is any one of these analogues, $Ff(n)$ terminates after a finite number of terms. If m is a real or complex number, mf is a scalar product as in (1.7), and $(mf)_s$ is the scalar product $m^s f_s$:

$$Smf=\sum_{s=0}^{\infty} \frac{(-1)^s m^{2s+1}}{(2s+1)!} f_{2s+1}, \qquad Cmf=\sum_{s=0}^{\infty} \frac{(-1)^s m^{2s}}{(2s)!} f_{2s},$$

4. *Examples.* A few of the simplest well known functions will suffice to illustrate some of the preceding general theorems. The functions P, T were used (in another notation) by Sylvester;[6] λ, μ, θ were introduced by Liouville, Mobius, and Dirichlet, respectively; κ is Liouville's; σ, τ, ϕ are Euler's; ϵ is Kronecker's delta in a notation adapted to its present use; ζ is implicit in Liouville's and Dedekind's inversion formula. Each of these functions can be essentially generalized in many ways. The product and sum in what follows refer to $i=1,\cdots, P(n)$;

$$n=\prod p_i^{t_i}, \quad n>1, \quad t_i>0,$$

is the decomposition of the positive integer n into a product of powers of distinct positive primes p_i, so that $P(n)$ is the number of distinct positive prime divisors of n. By convention, $P(1)=0$. The definitions of the remaining functions follow:

$$T(n)=\sum t_i; \quad T(1)=0.$$

$$\lambda(n)=(-1)^{T(n)}.$$

$$\mu(n)=(-1)^{P(n)} \text{ if } \prod t_i=1, \text{ or if } n=1, \quad =0 \text{ if } \prod t_i>1.$$

$\theta(n)=2^{P(n)}$, the total number of decompositions of n into a product of two coprime positive factors; $\theta(1)=1$.

$$\zeta(n)=1, \quad n\geq 1.$$

$\kappa(n)=1$ or 0 according as n is or is not the square of a positive integer.

$$\phi(n)=n\prod (1-p_i^{-1}), \text{ the totient of } n; \quad \phi(1)=1.$$

$$\sigma(n)=\prod (1-p_i^{t_i+1})/(1-p_i); \quad \sigma(1)=1,$$

the sum of the positive divisors of n.

$$\tau(n)=\prod (t_i+1),$$

the number of positive divisors of n; $\tau(1)=1$.

$$u(n)=n, \quad n\geq 1.$$

In the formulas below, p is a positive prime, and f^{-1} is a reciprocal as in (1.10). The following relations are immediate isomorphs, by the method generators, of trivial algebraic identities, such as

$$(1-x)^{-2}=(1-x)^{-1}(1-x)^{-1}:$$

$$\theta=\lambda^{-1}\mu^{-1}, \qquad \zeta=\mu^{-1}, \qquad \kappa=\lambda\mu^{-1}, \qquad \sigma=u\mu^{-1},$$

$$\tau=\mu^{-2}, \qquad \phi=u\mu, \qquad \mu\zeta=\eta.$$

Also from the same source,

$$L\mu(n)=-\epsilon(n,p^s)/s,$$

and the L functions of the remaining numerical functions are simply expressible in terms of $L\mu$:

$$L\lambda(n)=(-1)^{s-1}L\mu(n), \quad Lu(n)=-nL\mu(n).$$

From the above relations, such as $\theta=\lambda^{-1}\mu^{-1}$, the L functions of $\theta,\cdots$ are written down by (2.1) from those already derived. For example,

$$L\theta=-(L\lambda+L\mu); \quad L\theta(n)=[(-1)^s-1]L\mu(n)$$

whence, $L\theta(n)=2\epsilon(n,p^{2s+1})/(2s+1)$. Similarly for the others.

As an example of f_t (as in Section 2), $\zeta_t(n)$ is the total number of decompositions of $n(>1)$ into a product of t integers each >1; $\zeta_t(1)=0$; and from $\zeta=\mu^{-1}$,

$$L\zeta(n)=-L\mu(n)=\epsilon(n,p^s)/s.$$

Again, if t is a positive integer, it is seen from the generator of ζ that

$$\zeta^t=\prod {}_{-t}C_{t_i},$$

with C as in (2.1). Hence, by (2.1),

$$\sum_{s=0}^{\infty} {}_{\iota}C_s\zeta_{\iota} = \prod {}_{-\iota}C_{\iota_i}.$$

The numerical functions in this paper are functions of one variable. The combinatorial equivalent of the algebra of these functions refers to distributions into any given number of compartments which, without loss of generality, may be placed in a single row (or column) of a rectangular checkerboard. The theory of numerical functions of any finite number[7] of variables concerns distributions into all the compartments of the board.

REFERENCES, NOTES

1. E. Netto, *Lehrbuch der Combinatorik*, zweiten Auflage, 1927, section 176 and the references there given.

2. E. T. Bell, *An Arithmetical Theory of Certain Numerical Functions*, University of Washington Publications in Science, No. 1, 1915, 1-44; *Algebraic Arithmetic*, American Math. Soc. Colloquium Publications, Vol. 7, 1927; *An outline of a theory of arithmetic functions*, Jr. Indian Math. Soc., 17, 1928, 249-260, where further references are given. See also,

R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. American Math. Soc., 33, 1931, 579-662.

Other theories of numerical functions applicable to combinatorial analysis are those of

D. H. Lehmer, Trans. American Math. Soc., 33, 1931, 945-957; Bulletin of same, 37, 1931, 723-726; American Jr. of Math., 53, 1931, 843-854; 58, 1936, 563-572.

The suggestions for a "formal" theory of Dirichlet series in G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 1938, section 17.6 (pp. 250-252, and note, p. 258), were fully developed by Bell, 1915. The validation of the formal processes is by operations on one-rowed matrices, as in reference 5. The power of the method is missed unless account is taken of what was called (reference 5) E Multiplication; D multiplication applies in the present paper to Section 2.

3. E. T. Bell, Tohoku Math. Jr. 17, 1920, 221-231; greatly simplified proof, *ibid.*, 43, 1937, 77-78.

4. This is connected with the "ideal addition" of my 1915 paper (ref. 1), which was discarded in favor of addition as in (1.5). The connection between the two is given by (2.8). The distributive law follows from (11)(c), (2.8), (2.9); if (f), (g), (h) are restricted products,

$$(f)\,|(g)+(h)| = (f)(g)+(f)(g) = (fg)+fh).$$

The commutative law is obviously satisfied.

See also Vaidyanathaswamy, cited in reference 1.

5. E. T. Bell, *Euler Algebra*, Trans. American Math. Soc., 25, 1923, 135-154. Examples at end of paper.

6. He called them the multiplicity and the manifoldness of n.

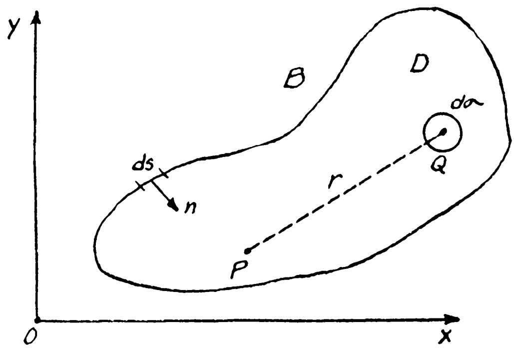
7. E. T. Bell, Bulletin American Math. Soc., 32, 1926, 341-345; Vaidyanathaswamy, reference 1.

California Institute of Technology.

On a Property of the Laplacian of a Function in a Two Dimensional Bounded Domain, When the First Derivatives of the Function Vanish at the Boundary

J. Kampé De Fériet

Let us consider, in the plane *oxy*, a *bounded domain* D, having for boundary a regular* closed curve B



We shall note by (C), (R), (H) and (L) the four following functional spaces:

(C) is the space of all functions $f(x,y)$ *continuous* in D;

(R) is the space of all functions $f(x,y)$ *regular* in D, that is:

$$\alpha)\ f, \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \text{ are continuous in } D+B$$

$$\beta)\ \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2} \text{ are continuous in } D$$

(H) is the space of all functions $f(x,y)$ *harmonic* in D, that is

$\alpha)$ f is regular in D

$\beta)$ f satisfies in D the Laplace equation:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

*What we mean here by *regular curve* is only that B fulfills the conditions required for the use of the Green formula.

It is obvious that (H) is a subspace of (R), and (R) a subspace of (C); (C) being itself a subspace of an Hilbertian real space, we can assume the ordinary definition for the scalar product of two functions $f(x,y)$, $g(x,y)$ belonging to (C):

$$(f,g)=\int_D f(x,y)\ g(x,y)d\sigma$$

and call f and g *orthogonal* if:

$$(f,g)=0.$$

As usual two subspaces are said to be orthogonal if each element f of the first subspace is orthogonal to each element g of the second subspace.

If we call (L) the subspace of (C) orthogonal to (H) it is obvious that the two subspaces (H) and (L) have only one common element, the null-function: $f(x,y)=0$.

Theorem I. If:

a) $\psi(x,y)$ *belongs to* (R)

b) $\psi=\text{constant}$ *on* B

c) $\dfrac{\partial\psi}{\partial n}=0$ *on* B

then

d) *the Laplacian** $\zeta(x,y)$ *of* $\psi(x,y)$:

$$\zeta=-\tfrac{1}{2}\,\Delta\psi$$

belongs to (L).

Theorem II. If:

a) $\psi(x,y)$ *belongs to* (R)

b) $\psi=\text{constant}$ *on* B

d) *the Laplacian* ζ *of* ψ *belongs to* (L)

then

c) $\dfrac{\partial\psi}{\partial n}=0$ on B.

In other words: *assuming that the conditions* a) *and* b) *are fulfilled the two statements* c) *and* d) *are equivalent.*

*We introduce here the numerical factor $-\frac{1}{2}$ to adopt the conventions of Fluid Mechanics.

Before proving theorems I and II let us make two remarks:

1) the statement b)+c) is equivalent to b′)

$$\frac{\partial\psi}{\partial x}=0 \qquad \frac{\partial\psi}{\partial y}=0 \text{ on } B$$

2) the statement a) implies that ζ belongs to (C).

The proof of theorem I is quite easy; let us start with the Green formula for two functions f and g belonging to (R);

$$\int_D (f\Delta g-g\Delta f)d\sigma=-\int_B \left(f\frac{\partial g}{\partial n}-\frac{\partial f}{\partial n}\right) ds;$$

let us apply it to:

$$f(x,y)=\psi(x,y) \qquad g(x,y)=v(x,y),$$

ψ belonging to (R) and V to (H); we get:

$$-\int_D V\Delta\psi d\sigma=-\int_B \left(\psi\frac{\partial V}{\partial n}-V\frac{\partial\psi}{\partial n}\right) ds$$

On account of b):

$$\int_B \psi\,\frac{\partial V}{\partial n}\,ds=\psi\int_B \frac{\partial V}{\partial n}\,ds=0$$

(by a classical property of every function V of (H)); and on account of c):

$$\int_B V\frac{\partial\psi}{\partial n}\,ds=0$$

Thus we get for every function V of (H)

$$\int_D V(x,y) \quad \zeta(x,y)d\sigma=(V,\zeta)=0.$$

In order to prove theorem II, let us consider the Poisson equation:

$$\Delta\psi=-2\zeta$$

ζ belonging to (L); if we assume only that ζ belongs to (C) the solution ψ of this equation belonging to (R) *and fulfilling condition* a) is given by the well known formula:

$$\psi(x_p,y_p)=\frac{1}{\pi}\int_D \zeta(x_Q,y_Q)G(x_p,y_p,x_Q,y_Q)d\sigma+k$$

k being the constant value of ψ on B

G being the Green function:

$$G(x_p,y_p,x_Q,y_Q)=\log\frac{1}{r}+U(x_p,y_p,x_Q,y_Q)$$

where U belongs to (H) as function either of (x_p,y_p) or of (x_Q,y_Q).
Thus according to d):

$$\int_D \zeta(x_Q,y_Q)\ U(x_p,y_p,x_Q,y_Q)d\sigma=0$$

for every point P in $D+B$.
With our assumptions a), b) and d) we get then:

$$\psi(x_p,y_p)=\frac{1}{\pi}\int_D \zeta(x_Q,y_Q)\ \log\frac{1}{r}\ d\sigma+k.$$

We can thus express the derivatives of ψ by:

$$\frac{\partial}{\partial x_p}\psi(x_p,y_p)=\frac{1}{\pi}\int_D \zeta(x_Q,y_Q)\frac{x_Q-x_p}{r^2}\ d\sigma$$

$$\frac{\partial}{\partial y_p}\psi(x_p,y_p)=\frac{1}{\pi}\int_D \zeta(x_Q,y_Q)\frac{y_Q-y_p}{r^2}\ d\sigma.$$

If the point P is on B $\dfrac{x_Q-x_p}{r^2}$ and $\dfrac{y_Q-y_p}{r^2}$ belong to (H) considered as functions of (x_Q,y_Q); then on account of d) the two integrals are equal to O and we get:

b′) $$\frac{\partial\psi}{\partial x}=0 \qquad \frac{\partial\psi}{\partial y}=0 \text{ on } B.$$

Having noted that b′) is equivalent to b)+c), we have thus proved that c) is true.

The statement b) being thus included in the final result b′) one might perhaps ask if it is not possible to suppress b) in the hypothesis of theorem II and to replace it by the more general assumption: $\psi(x,y)$ is continuous on B.

In fact the solution of the Poisson equation belonging to (R) is now given by:

$$\psi(x_p,y_p) = \frac{1}{\pi} \int_D \zeta(x_Q,y_Q) G(x_p,y_p,x_Q,y_Q)\, d\sigma + W(x_p,y_p)$$

W being the function (unique) belonging to (II) and taking the same values as ψ on the boundary:

$$\psi(x,y) = W(x,y) \quad \text{on } B;$$

our demonstration proves only that:

$$\frac{\partial\psi}{\partial x} = \frac{\partial W}{\partial x}, \qquad \frac{\partial\psi}{\partial y} = \frac{\partial W}{\partial y} \quad \text{on } B;$$

if we do not assume b) we can not prove b′).

It seems to me that the translation of theorems I and II in the language of Fluid Mechanics is worthwhile. Let us consider the two dimensional flow of an incompressible fluid in the bounded domain D; let $\psi(x,y,t)$* be the *stream function;* then the first derivatives of ψ give the components u and v of the *velocity* :

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} ;$$

the Laplacian ζ of ψ is the *vorticity* :

Theorem I′. *If at a given time t the stream function $\psi(x,y,t)$ belongs to (R) and if the velocity vanishes along the boundary B, then the vorticity ζ is orthogonal to every function V belonging to (II):*

$$\int_D V\zeta d\sigma = 0.$$

Theorem II′. *If at a given time t the stream function $\psi(x,y,t)$ belongs ro (R), if the boundary B is a stream line*† *and if the vorticity is orthogonal to every function V belonging to (II), then the velocity vanishes along the boundary B.*

Some consequences of theorem I′ give results, some of which are already known, but others seem new to me; for instance we can take for the harmonic function V :

$$1,\ x,\ y,\ x^2-y^2,\ xy$$

*We do not assume that the flow is permanent; in general ψ is dependent on the time; but all our computations being made at the same time t, t does not play a role in our results; it is only an auxiliary parameter in all the formulas.

†It means: ψ = constant on B; the velocity is tangent to B.

then we get for the vorticity ζ the following properties:

$$\int_D \zeta d\sigma = 0, \qquad \int_D x\zeta d\sigma = 0, \qquad \int_D y\zeta d\sigma = 0;$$

$$\int_D x^2\zeta d\sigma = \int_D y^2 d\sigma; \qquad \int_D xy\zeta d\sigma = 0$$

which have obvious mechanical significance. All these equations are particular cases of the following proposition; let us call the *moments* of $\zeta(x,y)$ of order $j+k$, the integral:

$$\mu_{j,k} = \int_D x^j y^k \zeta d\sigma \qquad j,k \text{ integers} \geqslant 0;$$

then there exist always two linear equations between the moments of a given order $m > 0$:

$$\mu_{m,0} - \frac{m(m-1)}{1.2}\mu_{m-2,2} + \frac{m(m-1)(m-3)(m-4)}{1.2.3.4}\mu_{m-4,4} - \cdots = 0$$

$$\mu_{m-1,2} - \frac{(m-1)(m-3)}{2.3}\mu_{-3,3} + \cdots = 0$$

as it is easily seen by taking for V the most general harmonic polynomial.

The theorems I′ and II′ are entirely independent of the physical properties of the incompressible fluid; they deal only with stream function, velocity, vorticity and not with mechanical properties (perfect or viscous fluid...). But, of course, if one has to solve a particular problem in Fluid Mechanics, for instance if one has to integrate the Navier-Stokes equations for a viscous fluid, the solutions must certainly satisfy our theorems. It is perhaps worth while to make this obvious remark, because, in order to find some particular classes of solutions, one has often made assumptions, consisting in the introduction in the computations, more or less arbitrarily, of a kind of "elementary" functions (Bessel functions and so on). Thus it is important to note that theorem I′ shows that harmonic functions are quite inadequate to represent the vorticity, when the velocity vanishes at the boundary of the domain.

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Functional Analysis in Topological Group Spaces*

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Introduction. The enormous activity in abstract groups on the one hand and general topology on the other has in recent years crystallized into a fruitful attitude whose main thesis is that the groups that occur frequently in analysis are, for the most part, those that can support a topology with respect to which the group functions xy and x^{-1} are continuous. Such groups are now called topological group spaces or briefly topological groups. Linear topological spaces, Banach spaces and Hilbert spaces are special topological abelian groups.

The classical theory of functionals, associated with such eminent names as Evans, Fréchet, Gateaux, Hadamard, F. Riesz and Volterra, can be seen in the light of modern investigations as a theory of functions whose arguments and values lie in a handful of special topological groups—function spaces of numerical continuous functions, of differentiable functions, and of the Lebesgue classes L and L^2. The main topics of the classical theory of functionals are (1) functional polynomials and functional power series expansions; (2) differential calculus of functionals; (3) functional differential equations with functionals as unknowns; (4) functional differential equations with numerical functions as unknown and numerous other types of linear and non linear functional equations.

By functional analysis in topological group spaces we shall understand the subject matter corresponding to topics (1)-(4) just mentioned. Now the theory of linear functions in Hilbert space and Banach space (see Stone's "Linear Transformations in Hilbert Space" and Banach's

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Note added January, 1947. This address was written in the summer of 1941. Since then, many mathematicians have urged the author to make this bibliographical exposition available in print. The author is glad to comply with this request. In spite of the war, many papers were published since 1941 on topological groups and functional analysis—the reader is referred to the reviews given in the review journal "Mathematical Reviews" under the headings of "Functional Analysis," "Groups" and "Topology." Subjects like "normed linear rings" have grown since 1941 into good sized mathematical domains. Several young mathematicians who are not even mentioned in the present paper have taken their place in the mathematical world and have published excellent papers on our subject since 1941. It is hoped that a supplementary bibliographical exposition will be written in the near future to cover the papers published since 1941.

"Opérations Linéaires") and the closely related subject of the modern integrals (Saks' "Theory of the Integral") are themselves vast mathematical domains that have been reviewed on many occasions and so we signalize them here without further comment. The abstract algebra and pure topology of topological groups themselves are systematically developed in Pontrjagin's "Topological Groups." The applications of functional analysis in linear topological spaces to general differential geometry with linear topological coordinates has been the object of an extended review (see my 1938 address before the American Mathematical Society in Bull. of Amer. Math. Soc., vol. 45, pp. 529-563 (1939) entitled "General Differential Geometries and Related Topics"). There is a considerable amount of pure functional analysis scattered throughout the papers on general differential geometry. We shall touch on some of these problems of pure functional analysis arising in general differential geometry.

We wish to emphasize here, once for all, that the major portions of modern functional analysis are not only generalizations of classical functional analysis which include classical functional analysis as a special case—important as that may be—but they are also generalizations whose novelty remains invariant under a passage to the special topological groups of classical functional analysis.

1. *Topological Differential Calculus.* We shall assume at the outset that the topology of a topological group is a Hausdorff topology. There is no loss of generality here in considering a Hausdorff topology instead of the Fréchet topology with its weaker separation axiom. By a *t. g. topological group G* we shall mean a topological group that satisfies the following additional postulate *P*: *given any element* $y \epsilon G$ *and a neighborhood U of the unit element of G, there exists a positive integer n and an element* $x \epsilon U$ *such that* $y = x^n$. If in addition, the group *G* is abelian (additively written) we shall speak of such a group as a *t. a. g. topological group G.* Every t. g. topological group is not connected nor is every connected topological group a t. g. Linear topological spaces, and hence Banach spaces, are t. a. g.'s.

A large number of differentials of functions with arguments and values in t. a. g.'s have been studied by me.*[37],[38],[39] Many of these differentials are in general distinct and several of them are equivalent to the Fréchet[8] differential only within equivalent Banach topologies whenever the t. a. g.'s are Banach spaces.[2] For the purpose of this brief review let us single out two of the more important types of differentials. Let $f(x)$ be a function with argument and values in t. a. g.'s and let $f(x)$ be defined in an open set containing the point x_0. A linear

*Superior numbers in parenthesis refer to the bibliography at the end of the paper function (i.e., additive and continuous $f(x_0;\,\delta x)$ of the independent t. a. g. variable δx is an M_1-differential of $f(x)$ at $x=x_0$ if, for some neighborhood N and function $\epsilon(x_0,x_1,x_2)$,

$$f(x_0+\delta x)-f(x_0)-f(x_0;\,\delta x)=\epsilon(x_0,\delta x,\delta x) \text{ for } \delta x\epsilon N$$

where the epsilon function is 0 for $x_1=0$, integer homogeneous in x_2, and has the following *uniformity property* with respect to some fixed neighborhood W: given a neighborhood V, there exists a neighborhood $U(V)$ such that $\epsilon(x_0,x_1,x_2)\epsilon V$ for $x_1\epsilon U(V)$, $x_2\epsilon W$.

Several fundamental theorems hold for M_1-differentials: (1) the uniqueness theorem; (2) continuity of $f(x)$ at $x=x_0$; (3) differentiability of the group sum of differentiable functions; (4) differentiability of the composition of differentiable functions; (5) theorem on the invariance of the differentiability property under topological isomorphisms of the t. a. g.'s with the corollary on the invariance of the differentiability property under a passage to equivalent Hausdorff topologies of the t. a. g.'s; (6) there exists a fixed neighborhood A with respect to which the following property holds: given a neighborhood V, there exists a neighborhood $U(V)$ such that $n[f(x_0+y)-f(x_0)-f(x_0;y)]\,\epsilon\, V$ for $y\epsilon U(V)$, $ny\epsilon A$, where n is any positive integer; (7) if the t. a. g.'s are Banach spaces, then M_1-differentiability is equivalent to Fréchet differentiability within equivalent Banach topologies.

It is of some interest to remark at this point that the characteristic postulate P for a t. a. g. topological group is used only in the proof of the uniqueness of an M_1-differential so that the remaining theorems hold good in topological abelian groups that are not t. a. g.'s.

The property in theorem (6) together with the linearity of $f(x_0;y)$ in y can be taken as the definition of a new type of differential, called a μ-differential. The above theorems (1)-(5) hold good for μ-differentials in t. a. g.'s. An M_1-differential is a μ-differential. It is an open question, however, whether a μ-differential is necessarily an M_1-differential. A noteworthy feature of the theory of M_1-differentials and μ-differentials is found in the pure topological algebraic character of the theory. By this we mean that the real number system does not enter into the theory and that the above theorem (5) on invariance holds.

If the t. a. g. topological groups are linear topological spaces, one can define a μ^*-differential whose definition is obtained by replacing the "positive integer" n in the definition of a μ-differential by the positive real number λ. It can be shown that, in linear topological spaces, μ^*-differentiability is equivalent to M_1-differentiability with an $\epsilon(x_0,x_1,x_2)$ function that is homogeneous in x_2 with respect to positive real multipliers.

Studies on nth order differentials in t. a. g.'s have been made with 1st order differentials taken to be either M_1-differentials or μ-differentials.

Millsaps[54] has just completed an interesting study of first order differentials of functions with arguments and values in t. g. topological groups. Millsaps' definitions of first order differentials reduce to the definition of an M_1-differential whenever the argument as well as the value groups are abelian.

Hyers[19] has shown that the notion of a pseudo-norm is sufficient to characterize linear topological spaces—this is a remarkable contribution to linear space theory. A pseudo-norm is a non-negative real-valued function defined for each element of a space and each element of a strongly partially ordered set. Banach space norm methods can be modified to develop a functional analysis in linear topological spaces. For example, Hyers has been able to generalize Fréchet's differential by defining and studying a Hyers differential[21] in linear topological spaces with the aid of a pseudo-norm. Recently Markoff has used Hyers' methods to treat the subject of the imbedding of topological spaces in topological groups.

LaSalle[25],[26] has solved a number of fundamental problems for linear topological spaces with the aid of a slightly modified Hyers' pseudo-norm. For example, he has obtained necessary and sufficient conditions on the space for the existence of a real-valued functional in a linear topological space; he has shown that the space of linear transformations in linear topological spaces can be topologized in such a manner as to form a linear topological space; and he has developed (independently of Hyers) the theory of a differential[26] which appears to be equivalent to Hyers' differential.

In his California Institute 1941 thesis, LaSalle has gone further and shown that a generalization of Hyers' pseudo-norm is fundamental to the study of topological spaces.[27],[28] He then applies the pseudo-norm methods to the development of functional analysis in certain spaces, say P_1-spaces,[27] that are generalizations of linear topological spaces. The multiplicative domain of LaSalle's P_1-spaces is a ring with valuation (norm) whose norm satisfies the inequality $\|\alpha\beta\| \leq \|\alpha\| \, \|\beta\|$. A t. a. g. is a P_1-space. However, there exist P_1-spaces that are not t. a. g.'s. LaSalle's theory of linear functions and its application to a differential calculus[27] in P_1-spaces is full of interesting results.

Finally we wish to point out that the first definition of a differential[52] in non-metrical spaces was given in 1936 by Michal and Paxson. Unfortunately, the Michal-Paxson differential has meaning only for functions whose arguments as well as values are in the *same* linear topological space. It is still an open question whether the differentiability theorem on the composition of differentiable functions is valid for this type of differential.

2. *Normed Rings.* In 1932, R. S. Martin and Michal studied[48] linear normed rings (with real or complex number multipliers) in connection with a generalization of the Fredholm integral equation theory. Infinite dimensional examples of linear normed rings were given with the property that for some elements α and β, $\|\alpha\beta\| \neq \|\alpha\| \, \|\beta\|$. Since then, Elconin, Michal, Hyers, Mewborn and Wyman have studied various topics in the theory of linear normed rings: theory of ideals,[41] power series expansions,[41],[43] differential equations,[41],[43] and applications to general differential geometry[33],[35] with Banach coordinates—especially to general Riemannian geometry[35] and general projective differential geometry.[35],[50] Recently many authors have announced results (frequently without proofs) on the topology of ideals in linear normed rings. I. Gelfand[11] and others have indicated briefly how ideal theory in linear normed rings can be applied to Fourier analysis and to its generalizations in topological groups.

3. *Abstract Polynomials and Power Series Expansions.* Fréchet[9],[10] began the subject of abstract polynomials in some special linear topological spaces. A function is a polynomial of degree n, according to Fréchet, if it is continuous and its $(n+1)$st, but not its nth, difference vanishes identically. Fréchet's main result is that a polynomial can be written uniquely as a sum of homogeneous polynomials.

In his 1932 California Institute thesis (not published), R. S. Martin[29] made important contributions to the theory of polynomials in Banach spaces and complex Banach spaces. According to Martin, $f(x)$ is a polynomial if it is continuous and if

$$f(x+\lambda y) = \sum_{i=0}^{n} \lambda^i h_i(x,y).$$

Martin showed that his definition is equivalent to Fréchet's in Banach spaces but not in complex Banach spaces. This led Ivar Highberg[14],[15] in 1936 to his interesting study of functions, called pseudo-polynomials that satisfy Fréchet's conditions in complex linear topological spaces. In 1932, 1933 Martin, Michal, and Clifford employed the theory of polynomials in some outstanding problems in functional analysis.[40],[48]

In his thesis, Martin initiated also the study of polars and modular properties of homogeneous polynomials. In the 1934 Studia Mathematica that appeared in 1936, Mazur and Orlicz rediscovered some of Martin's results on polars and made some interesting contributions of their own[30],[31] in Banach spaces and in more general linear spaces.

In 1938, Taylor[66] proved several important theorems on sequences of homogeneous polynomials in complex Banach spaces. For example, he proved that an everywhere convergent sequence of homogeneous polynomials of degree k has a limit which is itself a homogeneous polynomial of degree k and that the corresponding sequence of moduli is bounded. Recently Van der Lijn[67] has made an extensive study of polynomials in abelian groups and then used it to make a neat comparative study of polynomial theories in Banach spaces.

Again in his thesis, Martin[29] initiated the theory of abstract power series in complex Banach spaces and developed what one might call the "Weierstrass viewpoint" of analytic functions in complex Banach spaces. He made a large use of the modular properties of polynomials and proved many important theorems such as the term by term Fréchet differentiability (of all orders) of a convergent power series within its "sphere" of convergence. Power series in Banach spaces were also studied by Martin with the aid of different methods from those in complex Banach spaces. In 1932, Michal and Martin obtained the best possible results, under their postulates, in connection with the abstract power series that generalize the Fredholm functional expansions of integral equation theory.[48] In 1933, Michal and Clifford gave theorems on analytic implicit functions[40] in Banach spaces and complex Banach spaces.

In his 1936 California Institute thesis, Taylor[61],[62] developed the "Cauchy viewpoint" of analytic functions in complex Banach spaces. Taylor calls a function analytic in an open set D, if it is continuous and has a Gateaux differential at each point of D. Taylor[63],[66] has made noteworthy and extensive investigations in this subject on several occasions. He has also supplemented his work by introducing the abstract Cauchy-Riemann equations.[62],[64]

An introduction to a theory of polygenic functions[46] in normed complex couple spaces[53] was given recently by Michal, Davis and Wyman. A characterization of the "directional Gateaux differentials" for a large class of polygenic functions was one of the main contributions.

Pinney[59] has applied power series in Banach spaces to the solution of some problems in general differential geometry. Govurin[12] announced (without proofs) some interesting theorems on power series in Banach spaces.

4. *Abstract Differential Equations.* An existence and unqiueness theorem for first order "ordinary" differential equations in Banach spaces with initial conditions was demonstrated by Kerner[23] in 1932. Later Michal and Elconin obtained this theorem by different methods as a corollary of their more general theory.[41] In 1936, Hyers and

Michal studied second order "ordinary" differential equations with *two point* boundary conditions[43] in Banach spaces. Existence and uniqueness theorems were given, and in the case of the differential systems that occur in general differential geometry, the Fréchet differentiability of the solution $x(t,x_0,x_1)$ as a function of the real variable t and the abstract boundary values x_0,x_1 were also studied. More recently Hyers and Michal have included in their differential geometric papers several theorems in pure functional analysis. These theorems include some fundamental existence, uniqueness, and Fréchet differentiability theorems on the solutions of second order "ordinary" differential equations with *one-point* initial conditions[44],[45] in Banach spaces.

Existence and uniqueness theorems for differential equations[41] whose unknown functions have arguments as well as values in Banach spaces were given by Michal and Elconin in 1935 and 1937. Two general existence and uniqueness theorems were proved for completely integrable abstract "Pfaffian" differential equations $f(x;\delta x)=F(x.f(x),\delta x)$ in Fréchet differentials with one-point initial conditions—one theorem was "in the small" and the other "in the large." Several important additional theorems were also given in some specail Banach spaces. Mewborn and Michal have recently treated a new type of functional differential equation[50] which is not of the Michal-Elconin type.

Abstract Pfaffian differential equations play a fundamental role in general differential geometry[33],[35],[50] and in the Michal-Paxson-Elconin generalizations[33],[42],[51],[47] of the Lie theory of continuous transformation groups. The field is fertile and much important work remains to be done along this direction.

Pinney[60] has generalized the Michal-Elconin theory of abstract Pfaffian equations to the case in which the independent variable of the unknown function is a linear topological variable, and has succeeded in proving a differentiability theorem on the solution as functions of the initial parameters. Pinney made a large use of M-differentials[35],[36] (to be distinguished from M_1-differentials) in this investigation and in his partially completed thesis investigation on an abstract calculus of variations.

Hyers[16],[17] and Paxson[55],[57],[58] in their respective 1937 California Institute theses have proved interesting theorems on first order "ordinary" differential equations in some restricted linear topological spaces. Hyers employs a generalized Lipschitz condition while Paxson makes use of the Brouwer-Tychonoff fixed point theorem in linear topological spaces. Leray and Schauder made some interesting applications of topology to non-linear equations[24] in Banach spaces.

In conclusion, we wish to signalize the work of the American and Russian schools on functional analysis in partially ordered linear spaces.[5]

H. A. Arnold(1) made interesting contributions to this subject in his 1939 California Institute thesis (unpublished).

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COLLEGIATE ARTICLES

Original Papers Whose Reading Does Not Presuppose Graduate Training

Escalator Numbers

by Pedro A. Piza

If we define an *escalator number* A_n by the relation $A_n=\sum a_n =\prod a_n$, where A_n is the gradual sum of any number n of rational summands a_n, whose sum at any point must equal to their gradual product, such as for instance

$$A_4=3+\frac{3}{2}+\frac{9}{7}+\frac{81}{67}=3\cdot\frac{3}{2}\cdot\frac{9}{7}\cdot\frac{81}{67}=\frac{6561}{938};*$$

it is easy to see that for any arbitrary $A_1=a_1\neq 1$ we can get a_2 by the requirement $A_2=a_1+a_2=a_1a_2$. Hence $a_1a_2-a_2=a_1$ and

$$a_2=\frac{a_1}{a_1-1}=\frac{A_1}{A_1-1}.$$

Similarly

$$a_3=\frac{A_2}{A_2-1}$$

and in general

$$(1) \qquad a_{n+1}=\frac{A_n}{A_n-1}.$$

This is a recurrence formula which permits us to compute consecutively as many *summand-factors* a_n as we please, whose gradual sums or products are the successive escalators A_n.

If we let $a_1=x$ and call x the *base* of an escalator A_n, we can use the convenient notation $A_n)x$ to mean the escalator A_n to the base x. Then by repeated use of the recurrence formula (1) we can obtain alternatively

*We call these numbers escalator numbers, or simply escalators, because they can be *climbed* in n steps by summation or in one step of n factors by multiplication. The investigation presented in this introductory brief note was suggested by solution of Schell's problem E686 of the *Americam Mathematical Monthly*.

the first few *cobasic* summand-factors a_n and the first few *cobasic* escalators A_n in terms of the base x, which base can be any irreducible rational number $x=r/s$ where r and s are relatively prime, (with the only obvious exception of $x=1$), as follows:

$$A_1)x=a_1= x\ .$$

$$a_2=\frac{x}{x-1}\ .$$

$$A_2)x=\frac{x^2}{x-1}\ .$$

$$a_3=\frac{x^2}{x^2-(x-1)}\ .$$

In general if we let t_n be the numerator of any summand-factor a_n, and let d_n be its denominator, so that $a_n=t_n/d_n$, then $t_n=x^{2n-2}$, $d_n=x^{2n-2}-D_{n-1}$ where D_{n-1} is the denominator of the previous escalator A_{n-1}.

We can also let

$$A_n=\frac{T_n}{D_n}$$

and then $T_n=t_n{}^2=\prod t_m$.

$$D_n=\prod d_m\ .$$

Many interesting and easily proved theorems concerning escalators A_n and their summand-factors a_n can be derived. Some of them are:

I) The product of any two cobasic escalators A_n and A_{n+1} is equal to the square of the first one plus the second.

In symbols: $A_nA_{n+1}=A_n{}^2+A_{n+1}$.

II) The denominator d_{n+1} of any summand-factor a_{n+1} is equal to the product of the numerator t_n (of a_n), by its denominator, plus the square of the product of the denominators of all the summand-factors previous to a_n.

$$d_{n+1}=t_nd_n+(d_1d_2d_3\cdot\cdot\cdot d_{n-1})^2\ .$$

III) The denominator of any escalator A_{n+1} is equal to the product of the numerator of the previous cobasic escalator A_n by its denominator, minus the square of the denominator of A_n.

$$D_{n+1}=T_nD_n-D_n{}^2=T_nD_n-(d_1d_2d_3\cdot\cdot\cdot d_n)^2$$

IV) If A_{n-1} and A_n are any two cobasic successive escalators, then $(2A_{n-1}-A_n)$, 2 and (A_n-2) constitute a triplet of rational Pythagorean numbers.

$$(2A_{n-1}-A_n)^2+2^2=(A_n-2)^2 .$$

V) The sum of the cubes of two successive cobasic escalators is equal to the continuous product of the first one, by the second, by the second minus unity, and by the second one plus unity minus the first one.

$$A_{n-1}^3+A_n^3=A_{n-1}A_n(A_n-1)(A_n+1-A_{n-1}).$$

VI) The value of any escalator minus unity is equal to the reciprocal of its successive summand-factor minus unity.

$$A_n-1=\frac{1}{a_{n+1}-1} .$$

VII) The product of any two cobasic successive escalators divided by their sum is equal to the square of the first one divided by twice the first one minus unity.

$$\frac{A_nA_{n+1}}{A_n+A_{n+1}}=\frac{A_n^2}{2A_n-1} .$$

VIII) If m is any positive integer exponent whatsoever and A_n and A_{n+1} are two cobasic successive escalators, the value of the fraction $A_n^m/(A_n-1)^m$ will remain unchanged if we add A_{n+1}^m to the numerator and A_n^m to the denominator.

$$\frac{A_n^m}{(A_n-1)^m}=\frac{A_n^m+A_{n+1}^m}{(A_n-1)^m+A_n^m} .$$

Each of the above theorems can be readily verified by some substitution and proved by induction.

Many other curious relations can be easily demonstrated by the use of these escalator numbers, a particularly interesting one being:

IX) If x_1, x_2 and x_3 are the three distinct roots of the equation $x^3=1$, then $x_1^p+x_2^p+x_3^p=0$ is verified for all prime values of the exponent p other than 3.

Although this theorem (IX) is not new for it appeared as a problem in Chrystal's Algebra, it is interestingly simple to demonstrate by escalator relations and is left as an exercise to the reader.

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A Speedy Solution of the Cubic

by JOHN T. PETTIT

(At my request, the author has extracted this particular solution of the cubic from his more general treatment of the cubic and the general algebraic equation.—ED.)

Consider the cubic

1). $$y^3+py+q=0.$$

If p and q have the same signs, we change the signs of the roots, and consider only

2). $y^3 \pm |p|y \mp |q| = 0.$

Let $$y=\frac{|q|}{|p|}z.$$

Substituting, we get

$$z^3 \pm \frac{|p|^3}{|q|^2} z \mp \frac{|p|^3}{|q|^2} = 0$$

or $$\frac{z^3}{(1-z)} = \pm\frac{|p|^3}{|q|^2} = K.$$

Figure I gives the graph of K against z. It is seen from the curve that $-3<z<1.5$ corresponds to $-\infty<K<\infty$. Thus for any value of K, a value of z can be found, and a real root can be gotten from

$$y=-\frac{q}{p}z.$$

If the cubic be in the form

3). $x^3+bx^2+cx+d=0$

we merely determine $\frac{p^3}{q^2}$ from the identities,

$$p=c-\frac{b^2}{3}, \text{ and } q=d-\frac{bc}{3}+\frac{2b^3}{27}, \text{ where } x=y-\frac{b}{3}.$$

The graph shows that for $K>-6.75$, only one real root exists, and for $K<-6.75$, three roots exist. (The discriminant verifies these facts.)

A simple rule for solving the cubic (3).

Compute $$\frac{p^3}{q^2}=K;$$

Take the corresponding z from the table; substitute this value in

$$x=-\frac{q}{p}z-\frac{b}{3};$$

reduce the equation to a quadratic and solve in the usual way. The remaining roots are thus:

$$x=\frac{qz\pm\sqrt{-4p^3-3q^2z^2}}{2p}-\frac{b}{3}=\frac{q}{2p}\left[z\pm\sqrt{-4K-3z^2}\right]-\frac{b}{3}.$$

Example:

$$x^3-x^2-2x+1=0$$

$$p=-2-\frac{1}{3}=-2.33, \qquad q=1-\frac{2}{3}-\frac{2}{27}=0.26.$$

Therefore, $$K=\frac{p^3}{q^2}=-189.$$

From the tables, $z=1.00$

and $$x_1=-\frac{q}{p}z-\frac{b}{3}=\frac{0.26}{2.33}+0.33=0.44$$

$$x=\frac{q}{2p}\left[z\pm\sqrt{-4K-3z^2}\right]-\frac{b}{3}$$

$$=\frac{(0.26)}{2(-2.33)}\left[1\pm\sqrt{-4(-189)-3}\right]+.33$$

$$x_2=-1.25, \quad x_3=1.80$$

which are correct to two decimal places.

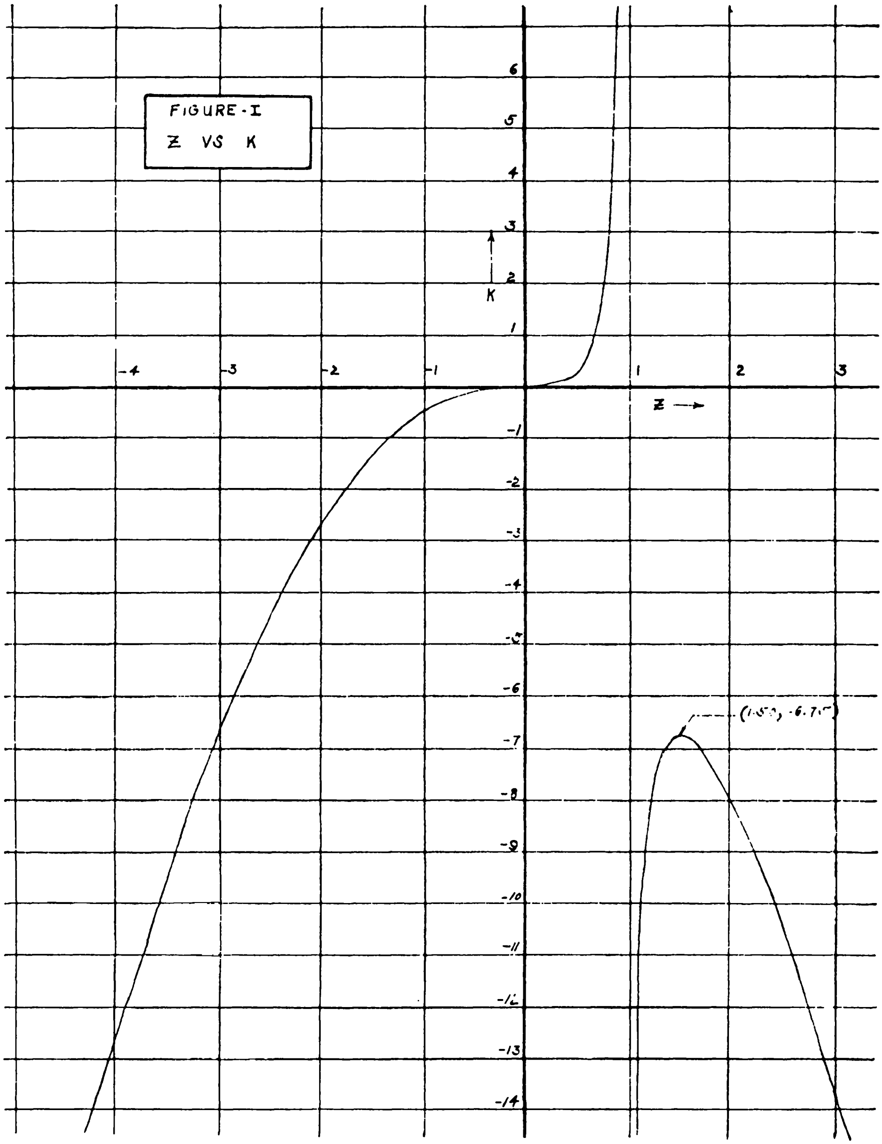


FIGURE - I
Z VS K
K
Z

TABLE I

Z	K	Z	K	Z	K	Z	K	Z	K
−3.00	−6.75	−2.47	−4.35	−1.94	−2.48	−1.41	−1.16	−0.88	−0.362
−2.99	−6.69	−2.46	−4.31	−1.93	−2.45	−1.40	−1.14	−0.87	−0.352
−2.98	−6.66	−2.45	−4.26	−1.92	−2.42	−1.39	−1.13	−0.86	−0.342
−2.97	−6.60	−2.44	−4.22	−1.91	−2.40	−1.38	−1.10	−0.85	−0.332
−2.96	−6.54	−2.43	−4.17	−1.90	−2.36	−1.37	−1.08	−0.84	−0.322
−2.95	−6.50	−2.42	−4.15	−1.89	−2.34	−1.36	−1.07	−0.83	−0.313
−2.95	−6.45	−2.41	−4.11	−1.88	−2.30	−1.35	−1.05	−0.82	−0.303
−2.93	−6.39	−2.40	−4.06	−1.87	−2.28	−1.34	−1.03	−0.81	−0.294
−2.92	−6.35	−2.39	−4.04	−1.86	−2.25	−1.33	−1.01	−0.80	−0.284
−2.91	−6.29	−2.38	−3.99	−1.85	−2.22	−1.32	−0.991	−0.79	−0.276
−2.90	−6.25	−2.37	−3.95	−1.84	−2.19	−1.31	−0.974	−0.78	−0.267
−2.89	−6.20	−2.36	−3.90	−1.83	−2.16	−1.30	−0.956	−0.77	−0.258
−2.88	−6.16	−2.35	−3.88	−1.82	−2.14	−1.29	−0.930	−0.76	−0.249
−2.87	−6.10	−2.34	−3.83	−1.81	−2.11	−1.28	−0.921	−0.75	−0.241
−2.86	−6.06	−2.33	−3.78	−1.80	−2.08	−1.27	−0.903	−0.74	−0.233
−2.85	−6.00	−2.32	−3.76	−1.79	−2.06	−1.26	−0.885	−0.73	−0.225
−2.84	−5.96	−2.31	−3.72	−1.78	−2.03	−1.25	−0.866	−0.72	−0.217
−2.83	−5.93	−2.30	−3.70	−1.77	−2.00	−1.24	−0.852	−0.71	−0.209
−2.82	−5.86	−2.29	−3.65	−1.76	−1.97	−1.23	−0.835	−0.70	−0.202
−2.81	−5.83	−2.28	−3.63	−1.75	−1.95	−1.22	−0.820	−0.69	−0.195
−2.80	−5.79	−2.27	−3.58	−1.74	−1.92	−1.21	−0.800	−0.68	−0.187
−2.79	−5.73	−2.26	−3.53	−1.73	−1.90	−1.20	−0.786	−0.67	−0.180
−2.78	−5.69	−2.25	−3.51	−1.72	−1.87	−1.19	−0.771	−0.66	−0.173
−2.77	−5.65	−2.24	−3.46	−1.71	−1.85	−1.18	−0.752	−0.65	−0.167
−2.76	−5.59	−2.23	−3.44	−1.70	−1.82	−1.17	−0.737	−0.64	−0.160
−2.75	−5.55	−2.22	−3.38	−1.69	−1.79	−1.16	−0.722	−0.63	−0.153
−2.74	−5.51	−2.21	−3.36	−1.68	−1.77	−1.15	−0.707	−0.62	−0.147
−2.73	−5.44	−2.20	−3.31	−1.67	−1.74	−1.14	−0.692	−0.61	−0.141
−2.72	−5.40	−2.19	−3.29	−1.66	−1.72	−1.13	−0.676	−0.60	−0.135
−2.71	−5.36	−2.18	−3.27	−1.65	−1.69	−1.12	−0.661	−0.59	−0.129
−2.70	−5.32	−2.17	−3.22	−1.64	−1.67	−1.11	−0.649	−0.58	−0.123
−2.69	−5.29	−2.16	−3.20	−1.63	−1.65	−1.10	−0.634	−0.57	−0.118
−2.68	−5.21	−2.15	−3.15	−1.62	−1.62	−1.09	−0.622	−0.56	−0.113
−2.67	−5.18	−2.14	−3.12	−1.61	−1.60	−1.08	−0.606	−0.55	−0.107
−2.66	−5.14	−2.13	−3.09	−1.60	−1.58	−1.07	−0.594	−0.54	−0.102
−2.65	−5.10	−2.12	−3.05	−1.59	−1.55	−1.06	−0.578	−0.53	−0.0975
−2.64	−5.06	−2.11	−3.02	−1.58	−1.53	−1.05	−0.566	−0.52	−0.0927
−2.63	−5.01	−2.10	−2.99	−1.57	−1.50	−1.04	−0.549	−0.51	−0.0881
−2.62	−4.97	−2.09	−2.95	−1.56	−1.48	−1.03	−0.537	−0.50	−0.0834
−2.61	−4.93	−2.08	−2.92	−1.55	−1.46	−1.02	−0.522	−0.49	−0.0792
−2.60	−4.89	−2.07	−2.89	−1.54	−1.44	−1.01	−0.513	−0.48	−0.0750
−2.59	−4.85	−2.06	−2.85	−1.53	−1.41	−1.00	−0.500	−0.47	−0.0708
−2.58	−4.80	−2.05	−2.83	−1.52	−1.39	−0.99	−0.488	−0.46	−0.0666
−2.57	−4.76	−2.04	−2.79	−1.51	−1.37	−0.98	−0.476	−0.45	−0.0629
−2.56	−4.72	−2.03	−2.76	−1.50	−1.35	−0.97	−0.464	−0.44	−0.0591
−2.55	−4.67	−2.02	−2.72	−1.49	−1.33	−0.96	−0.451	−0.43	−0.0555
−2.54	−4.63	−2.01	−2.70	−1.48	−1.31	−0.95	−0.440	−0.42	−0.0522
−2.53	−4.59	−2.00	−2.66	−1.47	−1.29	−0.94	−0.429	−0.41	−0.0490
−2.52	−4.54	−1.99	−2.64	−1.46	−1.26	−0.93	−0.416	−0.40	−0.0457
−2.51	−4.50	−1.98	−2.60	−1.45	−1.24	−0.92	−0.406	−0.39	−0.0426
−2.50	−4.46	−1.97	−2.58	−1.44	−1.23	−0.91	−0.395	−0.38	−0.0398
−2.49	−4.41	−1.96	−2.54	−1.43	−1.20	−0.90	−0.384	−0.37	−0.0370
−2.48	−4.39	−1.95	−2.51	−1.42	−1.18	−0.89	−0.373	−0.36	−0.0344

TABLE I—*Cont.*

Z	K	Z	K	Z	K	Z	K
−0.35	−0.0318	0.19	0.00847	0.73	1.44	1.25	−7.80
−0.34	−0.0293	0.20	0.00999	0.74	1.56	1.26	−7.69
−0.33	−0.0270	0.21	0.0117	0.75	1.69	1.27	−7.59
−0.32	−0.0248	0.22	0.0136	0.76	1.83	1.28	−7.50
−0.31	−0.0228	0.23	0.0158	0.77	1.99	1.29	−7.41
−0.30	−0.0208	0.24	0.0182	0.78	2.16	1.30	−7.33
−0.29	−0.0189	0.25	0.0208	0.79	2.35	1.31	−7.26
−0.28	−0.0172	0.26	0.0237	0.80	2.56	1.32	−7.19
−0.27	−0.0155	0.27	0.0270	0.81	2.80	1.33	−7.13
−0.26	−0.0140	0.28	0.0305	0.82	3.06	1.34	−7.09
−0.25	−0.0125	0.29	0.0344	0.83	3.36	1.35	−7.03
−0.24	−0.0111	0.30	0.0386	0.84	3.70	1.36	−7.00
−0.23	−0.00911	0.31	0.0432	0.85	4.10	1.37	−6.95
−0.22	−0.00868	0.32	0.0482	0.86	4.54	1.38	−6.92
−0.21	−0.00766	0.33	0.0536	0.87	5.07	1.39	−6.90
−0.20	−0.00667	0.34	0.0600	0.88	5.68	1.40	−6.85
−0.19	−0.00576	0.35	0.0660	0.89	6.41	1.41	−6.83
−0.18	−0.00494	0.36	0.0729	0.90	7.29	1.42	−6.81
−0.17	−0.00420	0.37	0.0804	0.91	8.38	1.43	−6.80
−0.16	−0.00353	0.38	0.0885	0.92	9.73	1.44	−6.79
−0.15	−0.00294	0.39	0.0972	0.93	11.5	1.45	−6.78
−0.14	−0.00240	0.40	0.107	0.94	13.8	1.46	−6.77
−0.13	−0.00195	0.41	0.117	0.95	17.2	1.47	−6.76
−0.12	−0.00154	0.42	0.128	0.96	22.1	1.48	−6.75
−0.11	−0.00120	0.43	0.139	0.97	30.4	1.49	−6.75
−0.10	−0.000909	0.44	0.152	0.98	47.0	1.50	−6.75
−0.09	−0.000669	0.45	0.166	0.99	97.0		
−0.08	−0.000474	0.46	0.180	1.00	∞		
−0.07	−0.000321	0.47	0.196	....	....		
−0.06	−0.000204	0.48	0.213				
−0.05	−0.000119	0.49	0.231	1.01	−103.		
−0.04	−0.0000616	0.50	0.250	1.02	−53.0		
−0.03	−0.0000262	0.51	0.271	1.03	−36.3		
−0.02	−0.00000784	0.52	0.282	1.04	−28.0		
−0.01	−0.000000990	0.53	0.317	1.05	−23.2		
0.00	0.000000000	0.54	0.342	1.06	−19.8		
0.01	0.00000100	0.55	0.370	1.07	−17.6		
0.02	0.00000816	0.56	0.398	1.08	−15.7		
0.03	0.0000278	0.57	0.431	1.09	−14.4		
0.04	0.0000667	0.58	0.465	1.10	−13.3		
0.05	0.000132	0.59	0.501	1.11	−12.5		
0.06	0.000230	0.60	0.540	1.12	−11.7		
0.07	0.000369	0.61	0.582	1.13	−11.1		
0.08	0.000556	0.62	0.627	1.14	−10.6		
0.09	0.000801	0.63	0.676	1.15	−10.1		
0.10	0.00111	0.64	0.728	1.16	−9.75		
0.11	0.00150	0.65	0.785	1.17	−9.41		
0.12	0.00196	0.66	0.846	1.18	−9.11		
0.13	0.00253	0.67	0.912	1.19	−8.89		
0.14	0.00319	0.68	0.982	1.20	−8.65		
0.15	0.00397	0.69	1.06	1.21	−8.42		
0.16	0.00488	0.70	1.14	1.22	−8.27		
0.17	0.00592	0.71	1.24	1.23	−8.09		
0.18	0.00711	0.72	1.33	1.24	−7.95		

University of California at Los Angeles.

CURRENT PAPERS AND BOOKS

Edited by
H. V. CRAIG

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

The purpose and policies of the first division of this department (Comments on Papers) derive directly from the major objective of the MATHEMATICS MAGAZINE which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy will be sent to the author of the original article for rebuttal.

Communications intended for this department should be addressed to

H. V. CRAIG, *Department of Applied Mathematics,*
University of Texas, Austin 12, Texas.

Essays in Science and Philosophy. By Alfred North Whitehead. Philosophical Library, New York, 1947. VIII+348 pages. Price $4.75.

This volume contains reprints of 23 separate papers, connected for the most part by little more than the fact that they come from the pen of one man. If it were not for the significant influence which this man has had upon contemporary thought, one would perhaps not attach a great deal of importance to this book. Chronologically the papers which it contains range from the year 1910 to the year 1941. They are grouped into four sections of approximately equal size.

Part I, carrying the subtitle Personal, consists of five papers, occupying 74 pages in all. The first three are biographical in the more narrow sense; the fourth is an essay on the influence of the geographical position on the character of the inhabitants of Southeastern England and of the other countries along the shores of the Straits of Dover and the North Sea. The last paper in this section, first published in the *Atlantic Monthly* for March, 1937, is concerned with the broad problems of the British Empire.

Part II, Philosophy (72 pages), contains six papers, of which one is a two-page statement on the philosophy of John Dewey; the others deal with topics such as one would expect to find under this subtitle: Immortality, Uniformity and Contingency. In part III, Education, there are again six papers, covering 74 pages. Among them is an address, entitled "Education and Self-Education" delivered in February, 1919 at the Stanley Technical School in Coventry.

Part IV, Science, occupies 116 pages divided over six papers, very heterogeneous in character. More than half of this section is taken up by three articles (Axioms of Geometry, Mathematics, Non-Euclidean Geometry), which can be found in volumes XI and XVII of the 11th edition of Encyclopaedia Britannica. There seems little justification for the reproduction of the text of these articles (the diagrams are omitted) available in a standard reference work, including even the bibliographies, now very much out of date, and the author's initials. Particularly out of place in this volume is the third of these articles, being written jointly with Bertrand Russell, whose initials B. A. W. R. appear along with A. N. W at the end of the article.

The first paper in Part IV, entitled "The first physical synthesis" is a delightful, brief discussion of the first century of modern physical science—Galileo, Newton, Bacon, Huyghens, Descartes. Perhaps the most valuable from the point of view of technical mathematical interest are the 18 pages devoted to the paper on "Indication, Class, Numbers, Validation", reprinted from the 1934 volume of Mind.

All in all, the volume aroused in the reviewer a feeling of distress, caused by its lack of coherence and urgency, by the recurring thought that the book does not do adequate justice to the author of "Science and the Modern World", of the "Introduction of Mathematics", of the "Adventure of Ideas", not to speak of the "Universal Algebra", nor of "Principia Mathematica". In spite of this rather disappointing total impression there have come moments of joy in the reading of the book, moments when bits of captivating frankness, of illuminating wisdom, of far-reaching vision projected out of the printed page the author's remarkable personality and the deeper causes of his influence upon the philosophers of the past decades. Perhaps quotation apart from the context will not destroy their significance for the reader of this review.

Recurring through several of the essays, we find the thoughts which are expressed in the following words on page 83: "The misconception which has haunted philosophic literature throughout the centuries is the notion of 'independent existence'. There is no such mode of existence; every entity is only to be understood in terms of the way in which it is interwoven with the rest of the Universe", and in these few lines on page 86: "The notion of the fixity of species and genera, and the notion of the unqualified definiteness of their distinction from each other, dominate the literary traditions of Philosophy, Religion and Science. Today, these presuppositions of fixity and distinction have explicitly vanished; but in fact they dominate learned literature. Learning preserves the errors of the past, as well as its wisdom. For this reason, dictionaries are public dangers, although they are necessities."

Although not occurring in papers on "Education", the following passages deserve to be thought over by the curriculum makers, the legislators for our schools and colleges: "The formal teaching at Cambridge was competently done, by interesting men of first-rate ability. But courses assigned to each undergraduate might cover a narrow range. For example, during my whole undergraduate period at Trinity, all lectures were on mathematics, pure and applied. I never went inside another lecture room. But the lectures were only one side of education. The missing portions were supplied by incessant conversation, with our friends, undergraduates or members of the staff. This started with dinner at about six or seven, and went on till about ten o'clock in the evening, stopping sometimes earlier and sometimes later. In my own case, there would follow two or three hours' work at Mathematics." (page 7) "Unless we are careful, we shall conventionalize knowledge. Our literary criticism will suppress initiative. Our historical criticism will conventionalize our ideas of the springs of human conduct. Our scientific systems will suppress all understanding of the ways of the universe which fall outside their abstraction. Our modes of testing ability will exclude all the youth whose ways of thought lie outside our conventions of learning. In such ways the universities, with their schemes of orthodoxies, will stifle the progress of the race, unless by some fortunate stirring of humanity they are in time remodeled or swept away." (page 26).

Mathematicians come in for a good bit of advice as well as for some encouragement. On page 98, we come across the following sentence: "Many mathematicians know their details but are ignorant of any philosophic characterization of their science." And on page 109, we find this remark: "If civilization continues to advance, in the next two thousand years the overwhelming novelty in human thought will be the dominance of mathematical understanding."

Of especial interest, in view of the importance attaching to modern sampling theory in various fields, are the following passages: ''The general problem is to examine, whether any isolated portion of our experience has any character which of itself implies a corresponding character, extending beyond the domain of that immediate example. In other words, we ask whether, on the ground of experience, we can deduce any systematic uniformity, extending throughout any types of entities, or throughout the relations between them." (page 132), and ''One of the dangerous fallacies in the construction of scientific theory is to make observations upon one scale of magnitude and to translate their results into laws valid for another scale. Almost always some large modification is required, and an entire inversion of fundamental conceptions may be necessary. ... I suggest that our sociological doctrines have made the same error in the opposite direction as to scales. We argue from small-scale relations between humans, say two men and a boy on a desert island, to the theory of the relations of the great commercial organizations either with the general public or internally with their own personnel." (pages 156,157).

This last quotation indicates that the author's interests extend into the field of the social sciences. An important task for them is suggested in a few lines from the essay on ''The study of the past'': ''Thus the interweaving of mass production with craftsmanship should be the supreme object of economic statesmanship. Here by craftsmanship I do not mean the exact reproduction of types of activity belonging to the past. I mean the evolution of such types of individual design and of individual procedure as are proper for the crude material which lies ready for the fashioning into particular products." (page 162).

His far-reaching influence in contemporary philosophy lends a special flavor and a particular importance to the pronouncements which are more directly related to that field: ''Philosophy is an attempt to express the infinitude of the universe in terms of the limitations of language" (page 14). ''What I am objecting to is the absurd trust in the adequacy of our knowledge. The self-confidence of learned people is the comic tragedy of civilization. There is not a sentence which adequately states its own meaning. There is always a background of presupposition which defies by reason of its infinitude". (page 95). ''But consciousness proceeds to a second order of abstraction whereby finite constituents of the actual thing are abstracted from that thing. This procedure is necessary for finite thought, though it weakens the sense of reality. It is the basis of science. The task of philosophy is to reverse this process and thus to exhibit the fusion of analysis with actuality. It follows that Philosophy is not a science." (page 113). ''The besetting sin of philosophers is that, being merely men, they endeavor to survey the universe from the standpoint of gods. There is a pretense at adequate clarity of fundamental ideas. We can never disengage our measure of clarity from a pragmatic sufficiency within occasions of ill-defined limitations. Clarity always means 'clear enough'." (page 123) ''I do not like this habit among philosophers, of having recourse to secret stores of information, which are not allowed for in their system of philosophy. They are the ghosts of Berkeley's 'God' and are about as communicative." (pages 143,144).

I do not know any better way to bring this discussion of the book to a close than by means of a final quotation which impresses one as a confession of faith of the philosopher-scientist: ''Power follows wisdom, because nature unlocks its secrets to the wise and dowers the temperate with zest and energy. Wisdom should be more than intellectual acuteness. It includes reverence and sympathy, and a recognition of those limitations which bound all human endeavour." (page 169).

July 19, 1947. ARNOLD DRESDEN.

HISTORY AND HUMANISM

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Papers on the history of Mathematics *per se*, the part it has played in the development of our present civilization and its relation to other sciences and professions are desirable for this department.

Opportunities For Mathematically Trained College Graduates*

by I. S. SOKOLNIKOFF

The Chairman of your Program Committee asked me to act as a discussion leader in the symposium on the opportunities for mathematically trained college graduates. This symposium will consist of a series of short addresses which I hope will be followed by an open forum so that the problems may receive more than one-sided illumination.

I should like to begin by gazing at the crystal ball and making some guesses as to what is likely to happen a year or two from now when the G.I. benefits run out, and when the influx of veterans is no longer with us, and how that is likely to affect the present sellers' market in mathematics. It is certain that two or three years from now, perhaps even sooner, there will be a period of contraction in the university attendance, and this should make available some mathematically trained individuals, now engaged in teaching, for positions in industry and government. Thanks to the short-sighted draft policies we are experiencing a severe deficit in mathematically trained personnel, and it is my guess that this deficit will be with us for many years to come because the new channels for absorption of mathematicians developed during the past five years would utilize whatever surplus of mathematically-trained personnel might be released from academic positions.

Prior to 1940 practically the only outlet for the employment of mathematicians, other than the statisticians and actuaries, was in the field of education. It was reliably estimated that in 1940 there were only 150 mathematicians employed by industry exclusive of actuaries and statisticians. The new channels for absorption of mathematicians in industry and government were definitely discernible even as early as 1930, and the rate of absorption was so greatly increased just before the

*This address is a part of a symposium on the subject held at Pomona College, Claremont, California, by Southern California Section of the Mathematical Association of America on March 8, 1947. It is printed here at the request of the Editors.

war that the demand for mathematicians in industry in 1940 so greatly exceeded the supply that the National Research Council saw fit to devote some 20 pages to this problem in its report to President Roosevelt. A section of this report* entitled "Industrial Mathematics" was distributed to all members of the Mathematical Association and the American Mathematical Society. I think it will bear careful rereading now since it appears to me to be even more timely at this moment than when it was presented to the President six years ago. I should like to make some speculative estimates which I believe are not very wide of the mark, and which will indicate that the colleges and universities should start recognizing now that, before very long, for every mathematics graduate employed in teaching there will be at least one employed by the government and industry. This means that we should start thinking of our students, not only as future teachers of mathematics, but also as potential research workers and consultants in industry.

I presume that Dr. Bollay and other speakers on this program will have something to say about the role played by the mathematicians in industrial and governmental research, and their remarks should give us indication of the type of revisions or additions to the mathematical bill of fare offered by the colleges. I also hope to make some remarks in this connection, but let me get back to my rough estimates.

While I made no actual count, I should like to guess that the membership list of the American Mathematical Society alone contains at least 400 names with industrial and governmental addresses. I arrived at this figure simply by the expedient of doubling the number of individuals with industrial addresses listed in the 1940 catalogue of members of the Mathematical Society. There are at least half as many more who are not affiliated with organizations whose primary aim is research in pure mathematics, so that this gives me a total of 600, which is 20% of the present membership of the Mathematical Society. This is not an insignificant percentage, and I believe you will agree with me that any estimates are decidedly in the nature of a lower bound. I have not included in these estimates the statisticians and actuaries, whose training is primarily mathematical, and there are 3500 members of the American Statistical Association alone. Of course, some names are duplicated, but I think that 3000 would present a very conservative guess for the number of mathematically-trained persons employed outside the collegiate profession. I should guess that the number in the latter is around 6000, which is precisely twice the number of individuals I estimate outside the collegiate field. While one might argue that we have reached the peak in the employment of teachers of mathematics, there

*_Research—A National Resource-II_, Section VI, Part 4, pp. 268-288, A House Document, 77th Congress.

is no indication that we will be able to satisfy the demands of industrial and governmental organizations in the foreseeable future.

Now a word about the training of mathematics majors. It is quite generally recognized now that we have not paid enough attention in the past to the problem of training in several important fields of mathematics. Prior to 1940 there were very few institutions where such disciplines as analytical dynamics, theory of relativity, hydrodynamics, potential theory, and elasticity were offered on a graduate basis by the mathematics departments. I can count on the fingers of my right hand those institutions where two or three of the subjects I mention were offered regularly by the mathematics departments and none at all that gave all of them. This deficiency was recognized some five years ago at Brown, New York University, and the Massachusetts Institute of Technology, which established some graduate work in several fields that go by the name of applied mathematics. Several other institutions are following their example, and I feel confident that before long this country will overcome this deficiency and will attain the same degree of preeminence in these fields that it has achieved in algebra, analysis, and geometry. I am making these remarks not because I think that the mathematician going into industrial work should have a special brand of training limited to what is called applied mathematics; what I have in mind is that training of all mathematics majors is essentially lopsided. How many of our colleges offer to undergraduates such mathematics courses as theoretical mechanics, potential theory, numerical computation, and the like? To my mind a teacher of mathematics is just as deficient in his training if he has had no contact with analytical mechanics as an industrial mathematician without a course in the theory of functions. My plea is not for a specialized technical training, but for training that leaves no important gaps unfilled.

In the time I have at my disposal I can only mention the subject of numerical analysis which, with the introduction of high-speed electronic and relay computers, is bound to leave a deep imprint on the development of all phases of mathematics. This field of mathematical computation, in my opinion, should be placed in the top priority group of offerings in our mathematics departments.

I think the industrial concerns would prefer to have men with broad training, a rounder training, rather than employ specialists in various fields of applied mathematics, because they recognize, more keenly than we do, Michael Faraday's dictum that there is nothing so prolific of utility as abstraction. This attitude is mirrored pretty well by our stronger engineering colleges which, to mix a metaphor, keep their ears close to industrial ground and are revising their curricula in the direction of broader training in theoretical subjects. The new generation of engineers is conscious of the advantages of an axiomatic approach to practical matters, and this is one of the reasons why we find in many of our graduate courses in mathematics more engineers and physicists than mathematics majors.

It must be admitted that the mathematician, to the detriment of mathematics as a whole, has done little to cultivate the new type of clientele, and tended to keep aloof from problems facing his non-mathematical colleagues. As a result we have a wasteful duplication and profusion of mathematics courses offered by such departments as Engineering, Physics, Economics, Psychology, Genetics, and so forth. Since the problem of providing suitable training in applied mathematics admittedly cannot be solved by non-specialists, many of our forward-looking institutions are appointing mathematicians as professors of engineering. It remains to be seen whether mathematicians would find the climate in other departments conducive to productive mathematical work, but it must be admitted that this "back door" solution of the problem is better than no solution at all. It is high time that we got off the pedestal and reflected on what this means.

University of California at Los Angeles.

The Use of Mathematicians in the Aircraft Industry

by WILLIAM BOLLAY*

In order to describe the use of mathematicians in the aircraft industry it is well to obtain a perspective of a typical modern aircraft company. About 20% of the personnel is in engineering, the remaining 80% are in the factory and business offices. Most of the engineering design of an airplane is carried out on the basis of previous experience. Thus, most of the structural and aerodynamic designs of wings, fuselages, powerplant installations, etc., is carried out on the basis of empirical rules and the application of relatively elementary engineering principles. Whenever the designer departs from previously established practice, he proceeds to test the component under development either full scale or on a model scale. Thus the designer ordinarily builds a wind-tunnel model of the complete airplane and by empirical modifications develops the stability, control, and performance characteristics which he desires.

*Director, Aerophysics Laboratory, North American Aviation, Inc.

(This paper was a part of the symposium mentioned in the previous article by Professor Sokolnikoff.—ED.)

Similarly the structural designer builds models or full scale duplicates of wing panels, etc., which he tests under the simulated loading conditions. Until a few years ago practically all airplanes were designed and built in this manner—relying largely upon component test to verify the estimates of the designer. This technique has on the whole been remarkably successful. Recently, however, as airplane speeds have become higher and higher a number of difficulties have arisen with this technique. First of all it has become extremely expensive to carry out the extensive model tests. An elaborate wind-tunnel model for test in a high speed wind-tunnel may cost as much as $100,000.00. Secondly this empirical process of trial and error has often failed to indicate the fundamental nature of the physical phenomena and thus much time and money have been wasted in testing hundreds of models without learning the general laws which would permit safe extrapolation in the future.

Finally it has been found that some of the problems of the airplane designer defy model testing. This is particularly true for high speed flight in the vicinity of the velocity of sound. Thus it has been found necessary to rely on full scale flight testing for carrying out the final airplane devolopment. This is extremely expensive and hazardous. In view of these difficulties the aircraft engineering organizations have recently found it desirable to expand their research deparments, and to launch parallel analytical investigations on the more difficult problems. It is largely in connection with these analytical research investigations that the mathematicians have found their principal application.

The research section is ordinarily a small percentage of the engineering department (about 10%) and only about 5% of this section are mathematicians. Thus an engineering department of 2,000 engineers may include about 10 mathematicians, about half of these having a Ph. D., the others having an M. S. or equivalent. This is about the ratio in our company. I believe that most other companies if anything, have somewhat smaller percentages.

Before explaining in detail the mathematical problems encountered in aeronautical research, I should like to digress and point out that within the aircraft industry, aeronautical research is the principal use of mathematicians as mathematicians. There are however, also other possibilities. Many mathematicians have during the war developed into competent aeronautical engineers and have worked on all phases of airplane design, experimentally as well as analytically. This possibility is open to almost any young college graduate in mathematics who has also a good basic training in physics. If the young mathematician with a BA or BS has little supplementary training in physics or engineering, he may find a position in the aircraft industry as a computer or supervising a group of computers; however, his possibilities for advancement are limited. For he does not ordinarily know sufficient mathematics to be useful in solving problems of applied mathematics; nor does he know sufficient physics or engineering to make a competent engineer.

The senior mathematician who wishes to engage in aeronautical research must have a good working knowledge of classical mathematics. It is very desirable that he have in addition a good working knowledge of classical physics, particularly mechanics, hydrodynamics and electricity and magnetism. Unless the mathematician has this intimate working knowledge of physics he will not be able to set up and solve his own problems. For ordinarily the actual engineering problems encountered are too difficult to be capable of exact solution. It therefore requires a good physical insight into the problems to know what approximations are permissible in formulating the mathematical problem and in carrying through its solution.

The types of problems which have been and are being worked out very successfully by mathematicians together with aeronautical engineers are as follows:

(1) Flutter and vibration of airplanes, propellers, and engines, and analysis of servo systems.

Required mathematics:

(a) Solution of a system of linear differential equations with constant coefficients.

(b) Matrix algebra.

(c) Operational calculus and Laplace transforms.

Desirable related background:

(a) Aerodynamic theory of non-stationary forces on a vibrating wing.

(b) Solution of circuit equations in electrical net works.

(2) Pressure distribution around airfoils.

Required mathematics:

(a) Conformal mapping.

Desirable related background:

(a) Potential theory.

(3) Lift distribution over wings.

Required mathematics:

(a) Solution of an integral equation by expansion of unknown into Fourier Series.

(4) Supersonic aerodynamics.

Required mathematics:

(a) Solution of partial differential equations by the method of characteristics.

Desirable related background:

(a) Knowledge of Theory of Sound including particularly solutions of the wave-equation.

(5) Theory of elasticity.

Required Mathematics:

(a) Solution of ordinary and partial differential equations.

(6) Heat Transfer.

Required mathematics:

(a) Solution of the potential and heat conduction equations.

These examples may illustrate the mathematical problems encountered in aeronautical research.

Another point which may be of general interest is the wage and salary level in the aircraft industry. The general salary structure in the aircraft industry was established by the U. S. Treasury Department during the war when most aircraft companies worked on government contracts. In general the salaries are similar to those paid in the Civil Service. Thus, for example, the following categories would be likely to apply to mathematicians.:

Job Description	*Salary Range*	*Minimum Experience Required*	*General Background*
Mathematician B	$1.20-1.40 per hour	Normally 2 years training in advanced college math	
Mathematician A	$1.40-1.65 per hour	4 years training and experience including 3 years college math	
Research Analyst B	$1.35-1.55 per hour	4 years training and experience including 4 years college or equivalent	BS, MS or MA
Research Analyst A	$305.-435. per month	6 years training and experience including 4 years college	BS, MS or MA
Research Engineer	$430.-610. per month	8 years training and experience including 4 years college	MS or Ph.D.

In conclusion I should like to summarize my report as follows:

(1) There are excellent opportunities now in the aircraft industry for a few mathematicians with experience in classical mathematics and physics and an interest in their applications.

(2) If the new mechanical and electrical computing devices come up to their expectations, the aircraft industry will undoubtedly offer even further opportunities to skilled mathematicians familiar with the use of these devices. For present day aircraft and missile testing is becoming so expensive that the industry will quickly take advantage of any savings by the use of these computing devices.

PROBLEMS AND QUESTIONS

Edited by
C. G. JAEGER and H. J. HAMILTON

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California.

SOLUTIONS

Following are the solutions to a few problems that were submitted to the former National Mathematics Magazine. The numbers are those under which the problems originally appeared:

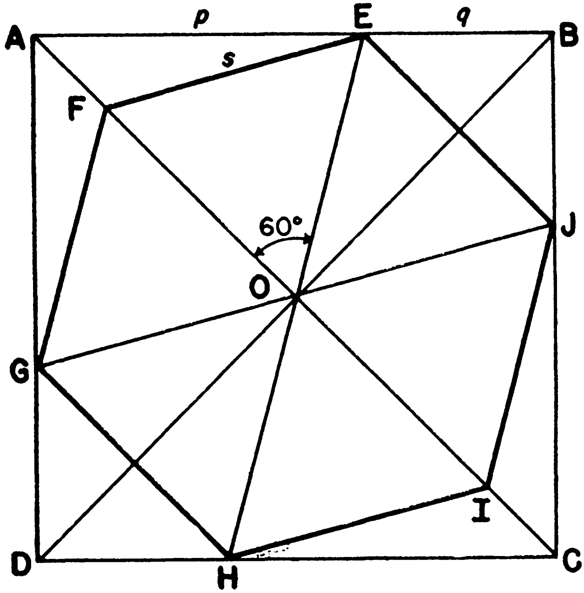
No. 610. Proposed by *Cleon C. Richtmeyer*, Central Michigan College, Mount Pleasant, Michigan.

Construct the largest possible regular hexagon that can be cut out of a given square $ABCD$ of side a, and compute the length of its side.

Solution by the *Proposer.*

The lines EH, GJ passing through the mid-point O of the diagonal AC and making angles of 60° with AC meet the sides AB, AD, DC, CB in the vertices E, G, H, J of the required hexagon $EFGHIJ$, the vertices F, I lying on the diagonal AG.

That the hexagon obtained is the largest possible may be seen as follows. If the hexagon is rotated to the right its diagonal GJ will decrease, while if it is rotated to the left the diagonal EH will decrease.



In the triangle AEF the angles opposite the sides AE and $EF=s$ are 120° and 45°, hence, by the law of sines,

$$AE : 3^{\frac{1}{2}} = s : 2^{\frac{1}{2}};$$

from the isosceles right triangle EBJ we have $EB = s/2^{\frac{1}{2}}$, hence, adding,

$$s = a \cdot 2^{\frac{1}{2}}/(3^{\frac{1}{2}}+1) = a(6^{\frac{1}{2}} - 2^{\frac{1}{2}})/2.$$

It may also be observed that $AE : EB = 3^{\frac{1}{2}}$.

The problem arose in a class in Applied Mathematics.

Also solved by *Howard Eves* who observes that if it is not required that the hexagon shall be in one piece, the given square can be cut up into pieces which can be assembled into a hexagon, in which case

$$S^2 = 2a^2\sqrt{3}/9.$$

No. 613. Proposed by *V. Thebault*, Tennie, Sarthe, France.

In an isosceles tetrahedron the three lines joining the center of an escribed sphere to the vertices of the corresponding face are mutually perpendicular; and conversely.

Solution by *Howard Eves*, College of Puget Sound.

Numbers in brackets refer to articles in N. A. Court's *Modern Pure Solid Geometry.*

Direct Theorem. Suppose the tetrahedron $ABCD$ is isosceles. Let E be the excenter opposite vertex A, and let B', C', D' be the midpoints of the edges CD, DB, BC respectively. Now AE is a diameter of the circumsphere of $ABCD$ (313) and therefore is also a median of $ABCD$ (298). Therefore AE, BB', CC', DD' are concurrent and trisect each other. It follows that tetrahedron $EB'C'D$ is homothetic with tetrahedron $ABCD$, and is thus also isosceles. Therefore $EB'=C'D'=\frac{1}{2}CD$ and $\sphericalangle CED=90°$. Similarly $\sphericalangle DEB=\sphericalangle BEC=90°$.

Otherwise. Let E, F, G, H be the centers of the escribed spheres of $ABCD$ corresponding to the vertices A, B, C, D respectively. (An isosceles tetrahedron possesses four trunc but no roof escribed spheres (252). Then $EFGH$ is the twin tetrahedron of $ABCD$ (312). Therefore, the trihedral angles $E-BCD$, $F-CDA$, $G-DAB$, $H-ABC$ are trirectangular (291).

Converse Theorem. Conversely suppose $\sphericalangle CED=\sphericalangle DEB=\sphericalangle BEC=90°$. Then $EB'=\frac{1}{2}CD=C'D'$, etc., and $EB'C'D'$ is isosceles.

Let I be the incenter of $ABCD$ and O' the circumcenter of $EBCD$. Then plane ICD is perpendicular to plane ECD (since these planes bisect dihedral angle $A-CD-B$). Now $O'B'$ is perpendicular to face ECD (for B' is the circumcenter of this face), and therefore $O'B'$ lies in plane ICD. Similarly $O'C'$ and $O'D'$ lie in planes IDB and IBC respectively. This guarantees that O' coincides with I. But $O'E$ is a median of $EBCD$ (287). That is, AIE passes through the centroid M of face BCD.

Take A' on IE such that $MA'=2EM$. Then $A'E$, BB', CC', DD' are concurrent at M and trisect each other there. Therefore $A'BCD$ is homothetic with $EB'C'D'$, and is thus isosceles. Let E' be the excenter of $A'BCD$ opposite vertex A'. Then, by the direct theorem, $E'-BCD$ is trirectangular. Therefore E' coincides with E (for there is only one trirectangular tetrahedron having a given base BCD and having its vertex on a given side of this base), and planes ABC and $A'BC$, then, must coincide, each being the plane other than BCD through the edge BC and tangent to the sphere having center E and radius equal to the distance of E from face BCD. Hence A and A' coincide, and $ABCD$ is isosceles.

The converse proposition may be stated as follows. If a face of a tetrahedron (T) and the corresponding excenter are the base and the vertex of a trirectangular tetrahedron, the same holds for the other faces of (T), and (T) is isosceles.

L. M. Kelly, U. S. Coast Guard Academy, proved the direct theorem by observing that the external bisecting planes of the dihedral angles of an isosceles tetrahedron T are parallel to the respectively opposite edges (this is an immediate consequence of art. 300 of *Modern Pure Solid Geometry*), hence these bisecting planes form the parallelepiped (P) circumscribed about (T), and (P) is rectangular (*ibid.* art. 291).

No. 620. Proposed by *Edmund Churchill*, Rutgers University.

Establish the convergence or divergence of the harmonic series modified by (a) the omission of all terms whose denominators do not begin with the digit 9; (b) the omission of all terms in whose denominators the digit 9 appears.

I. Solution by *B. P. Gill*, College of the City of New York.

The series in (a) diverges because the sum of the block of 10^k terms from $1/(9\cdot 10^k)$ to $1/(10^{k+1}-1)$ exceeds $10^k/10^{k+1}=1/10$ for $k=0,1,2,\cdots$.

For two proofs of the convergence of the series in (b) see Polya and Szego, *Aufgaben und Lehrsatze aus der Analysis*, vol. I, Abschn. I, Losung 124, s. 176. Reference is there made to A. J. Kempner, *American Mathematical Monthly*, 21, 1914.

II. Solution by the *Proposer.*

In series (b), let S_k be the sum of the reciprocals of all k-digit numbers in which the digit 9 does not appear. There are $8\cdot 9^{k-1}$ such numbers. Hence there are $8\cdot 9^{k-1}$ terms in S. No term of S_k exceeds $1/10^{k-1}$ and therefore $S_k<8\cdot 9^{k-1}/10^{k-1}$. Thus we may write

$$\sum_{k=1}^{\infty} S_k<8\sum_{k=1}^{\infty}(9/10)^{k-1}.$$

Since the right member is a convergent geometric progression, the convergence of (b) is established.

The terms omitted from the harmonic series according to (b) must themselves therefore form a divergent series; i. e. the harmonic series modified by the omission of all terms in whose denominators the digit 9 does not appear is divergent. But now, series (a), containing all the terms just mentioned and many more, is clearly divergent.

See Frank Irwin, *A Curious Convergent Series*, American Mathematical Monthly, 1916, pp. 149-152. Note that the entire harmonic series written in the scale of 9 has precisely the same appearance as (b), the divergence or convergence depending upon the interpretation of the symbols.

No. 622. Proposed by *V. Thebault*, Tennie, Sarthe, France.

Form a square of eight digits which is transformed into a second square upon increasing by unity the second digit from the left.

Solution by *Henry E. Fettis*, Dayton, Ohio.

Let y be a number such that its square is a number of eight digits differing by 10^6 from the square of a second number, x. Then

$$10{,}000 < y < 3{,}162$$

and $x^2 - y^2 = 10^6$ or

$$(x+y)(x-y) = 10^6.$$

Choosing various combinations of the prime factors of 10^6 as possible values for $x+y$ and $x-y$, it is found that only three combinations yield values of y between 10,000 and 3,162. They are

$x+y=5^4\cdot 2^4$	$x=5050$	$x^2=25{,}502{,}500$
$x-y=5^2\cdot 2^2$	$y=4950$	$y^2=24{,}502{,}500$
$x+y=5^5\cdot 2^2$	$x=6290$	$x^2=39{,}564{,}100$
$x-y=5\cdot 2^4$	$y=6210$	$y^2=38{,}564{,}100$
$x+y=5^4\cdot 2^5$	$x=10025$	$x^2=100{,}500{,}625$
$x-y=5^2\cdot 2$	$y=9975$	$y^2=99{,}500{,}625.$

The last of these satisfies the mathematically equivalent conditions of the problem, but it is not a solution in the strict sense, since the first two digits of y^2 must be considered rather than the second from the left, as stated in the proposal.

Also solved by *M. I. Chernofsky*, *C. S. Larkey*, *E. D. Schell*, *P. A. Piza*, and the *Proposer*.

MATHEMATICAL MISCELLANY

Edited by
MARIAN E. STARK

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: MARIAN E. STARK, Wellesley College, Wellesley 81, Mass.

Brief comments from emeritus professors on "what to retire to" would be of interest to readers. We all get there eventually if we live long enough. Tell us a few mathematical hobbies. Help us to look forward to something fascinating after "age 65".

What college or university in this country has the most students in mathematics at present? Send in within the next month the number at your institution for 1946-1947. The editor of this department hereby offers a Valuable Prize to the winner.

"A Manual of Operation for the Automatic Sequence Controlled Calculator", written by the Staff of the Computation Laboratory at Harvard, contains (in Chapter I) a history of digital calculators, (in Chapter II) a description of the Calculator, and a bibliography of computing methods and computing machines.

T. L. Smith, a former editor of this magazine, has resigned from Carnegie Institute of Technology and will remain with the Supersonic Wind Tunnels at Aberdeen Proving Grounds, as Chief Engineer.

The attention of the editor has been called to an Old-Babylonian tablet (1900-1600 B. C.), described on p. 38 of *Mathematical Cuneiform Texts*, O. E. Neugebauer and A. J. Sachs, New Haven, 1945. It is concerned with Pythagorean numbers and is "the oldest preserved document in ancient number theory." There one finds values of the hypotenuse and the shorter leg for fifteen Pythagorean triangles. It is noted that the ratio of the hypotenuse to the longer leg "decreases almost linearly" in this set of triangles. The suggestion is made that the word "Pythagorean" in this connection might better be replaced by "Babylonian". The tablet "gives the final link which connects the different parts of Old-Babylonian mathematics by the investigation of the fundamental laws of the numbers themselves."

"Ralph H. Beard, of the New York Telephone Company, was announced, on January 27th, as recipient of the 1947 Award of the Duodecimal Society of America, an award made annually for outstanding service to mathematical research with special relation to the duodecimal number system.

Mr. Beard is Editor of *The Duodecimal Bulletin* and has published a number of articles in the field of duodecimal mathematics, among which is his 'The Do-Metric System.' This is a proposal for a unified system of weights and measures bearing close relation to the systems now in use but co-ordinated on the dozen base. Mr. Beard points out that proponents of the French metric system would scrap our English weights and measures, which have many advantages of divisibility, convenience, and long usage, in favor of an arbitrary system based upon counting by tens, which has poor divisibility and cannot satisfactorily handle the circle. His system he claims, would retain the practical advantages of our present units, and add all the conversion advantages of a unified system through basing its units on counting by dozens."

New York. THE DUODECIMAL SOCIETY OF AMERICA.

OUR CONTRIBUTORS

Eric Temple Bell, Prof. of Math., Cal. Tech., was born in Aberdeen, Scotland, in 1883. Educated at U. of London, Stanford (A. B. 04), Washington (A. M. 08) and Columbia (Ph.D. 12), he joined Washington's faculty and remained there until 1927 when he was appointed to his present position. A prominent figure in mathematical circles for many years, Dr. Bell has received many honors, including the Bocher prize (24) and election to the National Acad. of Sci., for his researches in number theory. He has served as Pres., Math. Assn. and Vice Pres., A. M. S. and of Sec. A, A. A. A. S. Widely known as an author, he has written over twenty books, ranging from adventure stories to "Algebraic Arithmetic". A recent book, "The Magic of Numbers" will be reviewed in a forthcoming issue of this magazine.

J. Kampe de Feriet, Prof. at U. of Lille (Fr.), Hon. Dir., Inst. of Fluid Dynamics, was born in Paris in 1983. After studying at the Sorbonne (Doct. es Sci. 15) he was appointed Maitre des conferences (19) and Prof. (30) at Lille. Honorary memberships include Assoc. Fellow, R. A. S. (London) and Fellow, Inst. Aero. Sci. (N. Y.). His chief mathematical interests are fluid mechanics, statistical mechanics, mathematical statistics. Professor de Feriet has made several visits to the U. S. A. and recently returned to France after lecturing at Brown, Cal. Tech., Harvard and Michigan.

Aristotle D. Michal, Professor of Mathematics, California Institute of Technology, was born in Smyrna, Asia Minor in 1899, of Greek parents, and came to the United States at the age of twelve. He attended Clark University, (A. B., 1920; A. M., 1921) and then did additional graduate work at Rice Institute, where he received the degree of Ph.D. in 1924. After teaching at Rice Institute, Dr. Michal was awarded a National Research Fellowship for a two year period, spent at Chicago, Harvard and Princeton. His next position was an assistant professorship at Ohio State. In 1929 Professor Michal joined the faculty of the California Institute.

Dr. Michal has served on the Council of the American Mathematical Society and also as Associate Secretary of the Society. He has contributed many research papers to U. S. and foreign mathematical journals on subjects ranging from differential geometry to functional analysis and topological groups. In recent years he has also become interested in applied mathematics and is the author of a recent book, "Matrix and Tensor Calculus with Applications to Elasticity and Aeronautics."

Pedro A. Piza was born in Arecibo, P. R. in 1896. He attended the U. of Michigan and M. I. T. and returned to Puerto Rico in 1919, where he has been in business ever since. An amateur mathematician, mostly self taught, his special interests are number theory and geometry. Publications include articles in various periodicals and a book, "Fermagoric Triangles". Other hobbies are astronomy and philately.

John Pettit is a student at U. C. L. A.

Ivan S. Sokolnikoff, Prof. of Math., U. C. L. A., was born in Chernigov, Russia. in 1901. After attending the U. of Idaho (B. S. 26) and Wisconsin (Ph.D. 30) he joined Wisconsin's faculty, becoming Prof. of Math. in 1941, and came to U. C. L. A. in 1946. His chief interests lie in applied mathematics, especially elasticity theory. During the war he served in several U. S. research agencies (N. D. R. C., O. S. R. D., Hq. A. A. F.) as a consultant, official investigator, etc. Author of four books and contributor to various research journals, Dr. Sokolnikoff is also editor of the Quarterly Journal of Applied Mathematics and of a new series of books to be published by Wiley.

William Bollay, Dir. Aerophys. Lab., North Am. Aviation, Inc., was born in Stuttgart, Germany in 1911. He attended Northwestern (B. S. 33) and Cal. Tech. (M. S. 34, Ph.D. 36). After teaching aeronautics and applied mechanics at Cal. Tech, (36-37) and Harvard (37-41) Dr. Bollay served as a naval reserve officer in charge of research and development of gas turbines and jet propulsion engines. He rose in rank from ensign to commander. After the war he came to North American where he is directing guided missile research.